Proceedings

Cluj-Napoca MATh.en.JEANS Congress

Ariana-Stanca Văcărețu
(coord.)

Cluj-Napoca 2017

Cluj-Napoca MATh.en.JEANS Congress Proceedings

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Contents

Foreword

This volume contains the papers presented at the International MATh.en.JEANS Congress held in Cluj-Napoca, Romania, on April 7-8 2017. The Congress was organized by Colegiul Național Emil Racoviță Cluj-Napoca and the Faculty of Mathematics and Computer Science of the Babeș-Bolyai University Cluj-Napoca in partnership with the MATh.en.JEANS Association, the Cluj County School Inspectorate, the Romanian Mathematical Society, and the Samvs Rotary Club Cluj-Napoca.

The first MeJ Congress to take place in Romania brought together more than 230 participants: students, teachers and researchers coming from 14 high-schools from 9 cities in Romania, Italy and France, from two Romanian universities and two research institutes. The language of the congress was English. More information about the Congress can be found at mathinjeanscluj.weebly.com.

The congress was a unique moment for the high-school students who participated in the MATh.en.JEANS (MeJ) research workshops to share their research topic and their findings to both a wider scientific community, and the local community.
The congress provided an excellent opportunity to review and evaluate the accomplishments of research efforts since the beginning of the year. Its preparation allowed each student to become aware of the results achieved (by individuals, by one's own group and/ or the twin group) and to reorganise the new knowledge related to both the content and the process of the research. To present the research work and results, students needed to focus on their main results, order the stages of the research process, and find an appropriate way to reach the audience. Moreover, the students had to be prepared to receive the critical feedback of a wider scientific community.
More information about the MATh.en.JEANS project can be found at http://www.mathenjeans.fr/accueil.

These Proceedings contain the full text of the papers, in alphabetical order of the research topics. We are aware that, after the Congress, the students may have chosen to improve their research results and their articles; however, this publication aims to present the papers as they were shared in the Congress.
The authors are fully responsible for the content of the papers. However, we have to keep in mind that the authors are secondary school students and the articles may be affected by omissions and imperfections.

We sincerely thank every contributor for their time and creative effort. It is clear from the variety of the papers that the congress managed to attract many secondary school students who enjoy math research work.

We are especially grateful to the teachers and the researchers who facilitated the students' research and supported us in collecting the papers.

Ariana-Stanca Văcărețu
Coordinator of the Cluj-Napoca MeJ Congress

This article is written by students. It may include omissions and imperfections.

Adjusted Curves

2016-2017

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Research topic

Our research studies the geometric properties of a special type of closed curve as described by the following: *Each side of a regular polygon is divided into N regular segments. For each consecutive side, we construct the N segments* $[S_1S_n']$, $[S_2S_{n-1}']$, $[S_3S_{n-2}']$ ... *What can one say about the figure made by these segments?*

Results

At the end of our research we offered:

- a description of the curve’s general properties and of its behaviour when the number of divisions approaches infinity
- an equation for the characteristic dimensions and the surface of the curve
- the parametric equation of the curve's envelope
- an algorithm written in *JavaScript* that renders the curve based on two input parameters

Contents

<u>A. Geometry of the curve</u>

We first give the construction method, as presented within the 8-th topic:
Each side of a regular polygon is divided into N regular segments. For each consecutive side, we construct the N segments $[S_1S_n']$, $[S_2S_{n-1}']$, $[S_3S_{n-2}']$ ...
and the question of the problem:

What can one say about the figure made by these segments?
Two examples of construction are given below (figure 1) for a regular pentagon: the first for N=4 divisions and the second for N=20 divisions.

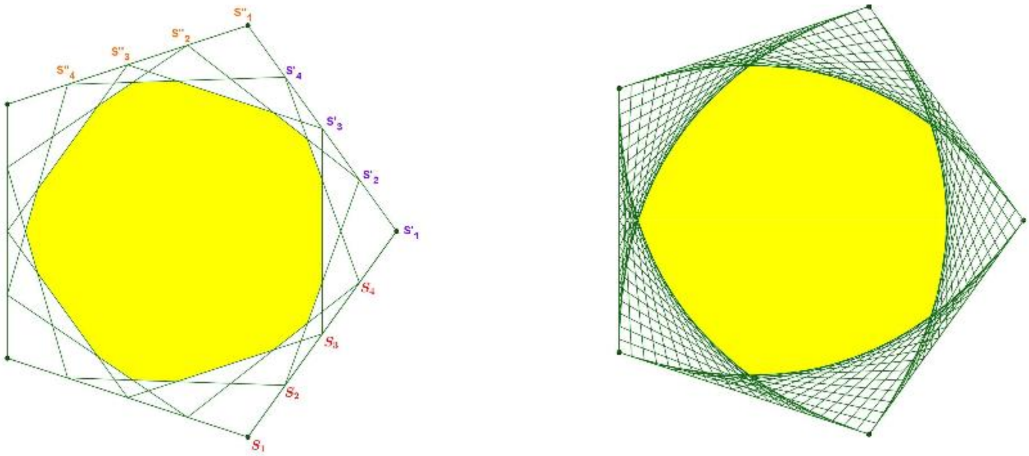
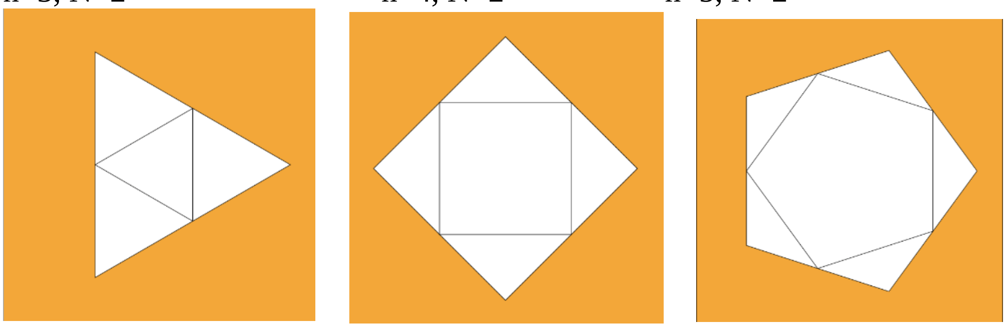


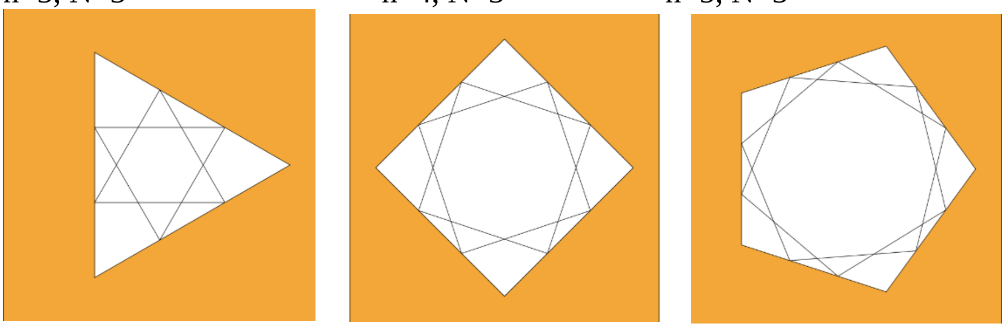
fig.1-Two sample constructions

We notice that the second curve looks smoother and that its sides resemble arcs of circle, whereas the first is just an irregular polygon with n(N-1) sides (n is the number of sides of the original polygon; here n=5, N=4).
For a better understanding of this process we give a comparative list of constructions for a triangle, a square and a pentagon for different values of N (figure 2):

n=3, N=2 n=4, N=2 n=5, N=2



n=3, N=3 n=4, N=3 n=5, N=3



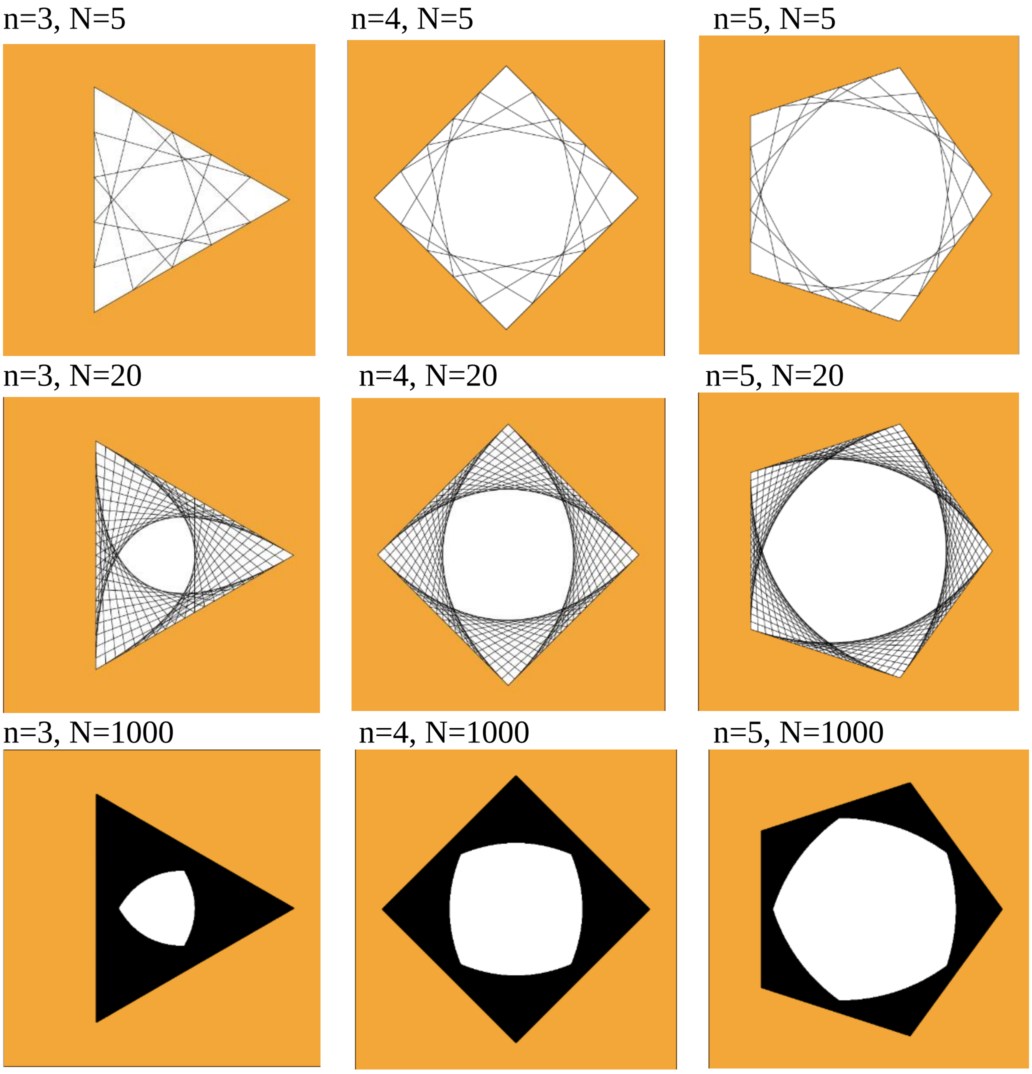


fig.2-The evolution of the figure as the number of divisions increases

It can be observed that whatever the number of sides of the original polygon, as the number of divisions approaches infinity, the curved formed at the inside of the polygon has arcs of circle as sides, namely:

- if n is odd, the curve is a Reuleaux polygon with n sides, of size equal to the radius R, where R is defined as the distance from a vertex of the curve to any point on the opposite arc
- if n is even, the curve is an "inflated" polygon with n sides arcs of circle of radius R, where R is defined as the distance from the middle of an arc to any point laying on the opposite arc

(These statements represent a combination of the first and second conjectures, proved in section C.)

We notice that we need to treat two distinct cases for n, depending on whether it is even or odd. Therefore, as we will see in section B. every equation will be different for the two cases.

B. Characteristic dimensions and surface

1. Odd number of sides

Before describing the structure quantitatively, we introduce the notations used later by means of a drawing (in this case a pentagon-figure 3):

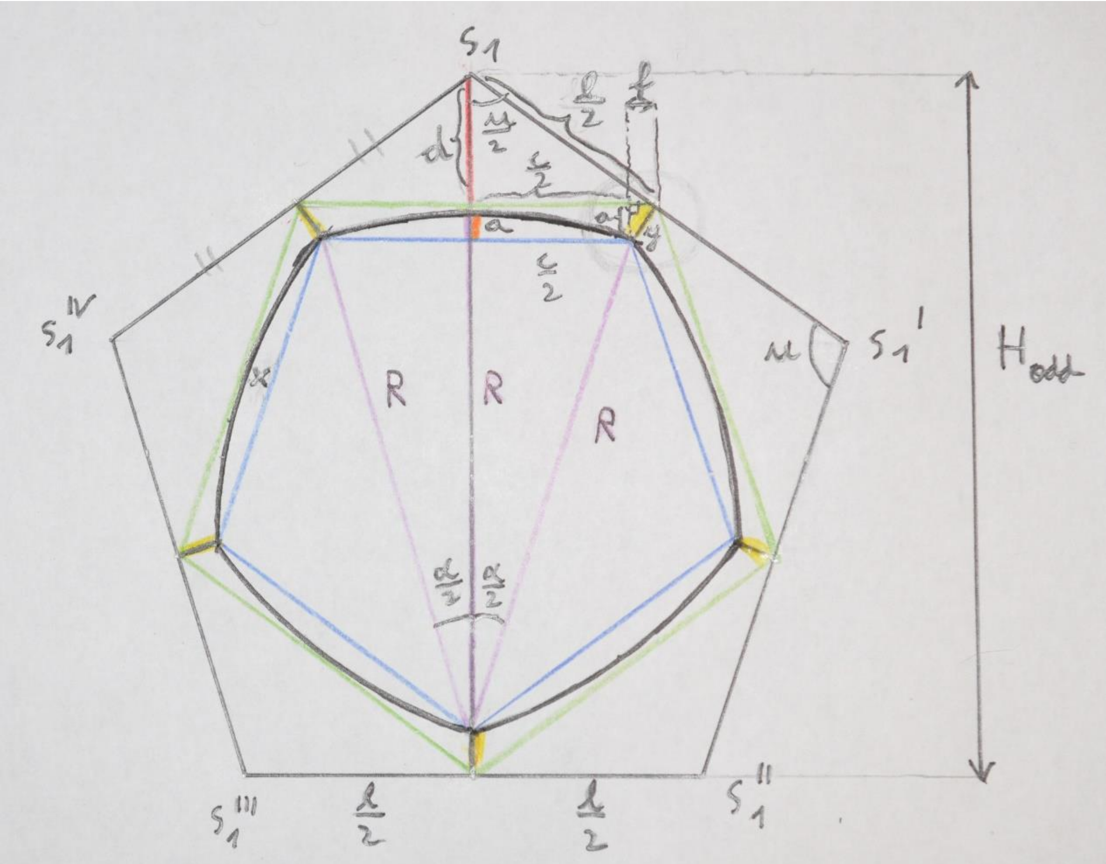


fig.3- The notations illustrated on a pentagon

-**l** is the side length of the original polygon (known)
-**u** is the angle measure of the original polygon (known)
-**n** is the number of sides of the original polygon (known)
-the black arcs are the sides of the Reuleaux polygon and their length is denoted by **x** (secondary unknown)
-the green lines are the middle lines of the polygon (the segments connecting the middle points of each 2 adjacent sides). These lines are tangent to the Reuleaux polygon in the middle point of each arc (Conjecture 4)
-the red lines are the distances from the vertices of the original polygon to the middle of the corresponding Reuleaux arc. Their length is denoted by **d** (intermediary unknown)
-the yellow lines are the distances from the middle of the sides of the polygon to the vertices of the Reuleaux polygon. Their length is denoted by **y** (intermediary unknown)
-the purple lines represent the radius of the Reuleaux polygon and their length is denoted by **R** (main unknown)
-**α** is the central angle of the Reuleaux polygon (main unknown)
-the blue lines are the chords of each arc and their length is denoted by **c** (secondary unknown)
-the little segment denoted by **a** represents the sagitta of chord **c**, or equivalently, the distance from the middle of the chord to the middle of the arc) (secondary unknown)

Using these notations, we intend to find the area of the Reuleaux polygon. The area is dependent (as seen later) only on the main unknowns **α** and **R**, which we first need to calculate.

<u>Calculation of **α_{odd}**</u>

For any regular polygon:

$$u = \frac{n-2}{n}180°$$

The height of the inner polygon with sides of length **c** is:

$$h = \frac{c}{2tan\frac{90°}{n}} = \frac{c}{2tan\left(\frac{u}{2(n-2)}\right)}$$

but:

$$h = R - a$$

whereas:

$$tan\frac{\alpha}{2} = \frac{\frac{c}{2}}{R-a}$$

From these relations, we get:

$$\boldsymbol{\alpha_{odd} = \frac{u}{n-2} = \frac{180°}{n-2}}$$

<u>Calculation of **R_{odd}**</u>

We first analyse a more detailed drawing (figure 4):

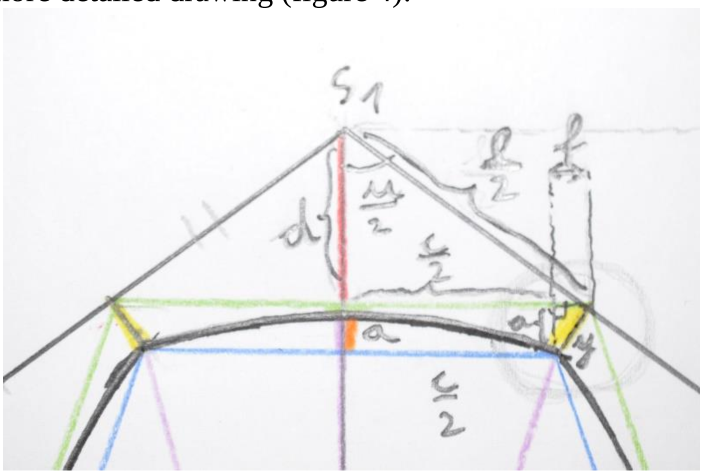


fig.4- Detail on figure 3

Looking at the picture and considering Conjecture 3 for odd-sided polygons ($R = H - d - y$, where H is the height of the original polygon) and Conjecture 4 (the middle lines are tangent to the arcs in their middle) we get the 8 following relations:

$$y^2 = a^2 + f^2 \; (\mathbf{1})$$

$$a = R\left(1 - \cos\frac{\alpha}{2}\right) \ (\mathbf{2})$$

$$\frac{c}{2} + f = \frac{l}{2} sin\frac{u}{2} \ (\mathbf{3})$$

$$\frac{c}{2} = R \ sin\frac{\alpha}{2} \ (\mathbf{4})$$

$$R = H_{odd} - d - y \ (\mathbf{5})$$

$$H_{odd} = \frac{l}{2}\cot\left(\frac{u}{2(n-2)}\right) (\mathbf{6})$$

$$d = \frac{l}{2} cos\frac{u}{2} \ (\mathbf{7})$$

$$\alpha = \frac{u}{n-2} \ (\mathbf{8})$$

Solving all these relations for **R**, we get a quadratic equation for **R**:

$$A\,\boldsymbol{R^2} + B\,\boldsymbol{R} + C = 0$$

With coefficients:

$$\boldsymbol{A = 1 - 2cos\frac{\alpha}{2}}$$

$$\boldsymbol{B = -l\left[sin\frac{u}{2}sin\frac{\alpha}{2} - \cot\left(\frac{u}{2(n-2)}\right) + cos\left(\frac{u}{2}\right)\right]}$$

$$\boldsymbol{C = \frac{l^2}{4}\left\{sin^2\frac{u}{2} - \left[\cot\left(\frac{u}{2(n-2)}\right) - cos\left(\frac{u}{2}\right)\right]^2\right\}}$$

And two distinct positive solutions:

$$\boldsymbol{R_{1,2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}}$$

However, **R** is unique and we need to find the right solution.
From the 8 relations above we can also derive that:

$$H_{odd} - d = \frac{-B}{2A}$$

but we already know that:

$$R = H_{odd} - d - y$$

and $y > 0$ (like any distance, **y** is positive)

Therefore, we need to choose the smaller solution, which gives:

$$R_{odd} = min(R_1, R_2)$$

2. Even number of sides

For this situation, too, we analyse a specific drawing (figure 5) with the same notations (here a square):

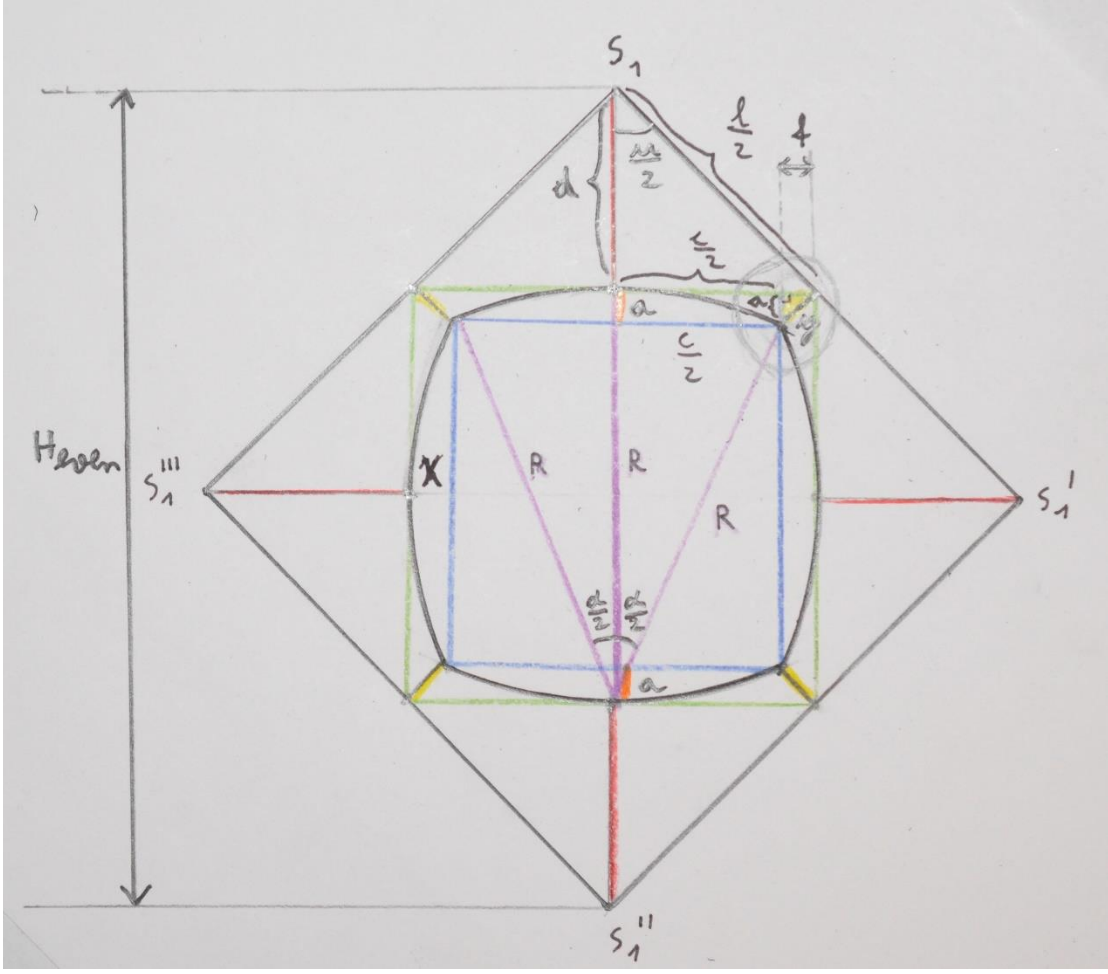


fig.5- The notations illustrated on a square

Calculation of $\boldsymbol{\alpha}_{\mathbf{even}}$

The relations needed are:

$$R - 2a = c \cdot tan\frac{u}{2} \ (\mathbf{1})$$

$$a = R\left(1 - cos\frac{\alpha}{2}\right) \ (\mathbf{2})$$

$$c = 2R\ sin\frac{\alpha}{2} \ (\mathbf{3})$$

From these relations, we get:

$$tan\frac{u}{2}sin\frac{\alpha}{2} - cos\frac{\alpha}{2} = -\frac{1}{2}$$

Solving this equation for $\frac{\alpha}{2}$ we find:

$$\boldsymbol{\alpha_{even} = 2\left\{\frac{u}{n-2} - \sin^{-1}\left[\frac{\sin\left(\frac{u}{n-2}\right)}{2}\right]\right\}}$$

<u>Calculation of **R**</u>

The relations needed are:

$$R = H_{even} - 2d\ (\mathbf{1})$$

$$H_{even} = \frac{l}{sin\left(\frac{u}{n-2}\right)}\ (\mathbf{2})$$

$$d = \frac{l}{2}cos\frac{u}{2}\ (\mathbf{3})$$

The first relation is the particular form of Conjecture 3 for even-sided polygons (note that **y** does not appear in this equation), and the second one is the length of the longest diagonal in an even-sided regular polygon. Notice that in the third equation **d** has the same expression as for odd-sided polygons. From the three relations and using the notation $\frac{1}{sin} = csc$ (cosecant function) we find:

$$\boldsymbol{R_{even} = l\left[\csc\left(\frac{u}{n-2}\right) - cos\frac{u}{2}\right]}$$

<u>Intermediary and secondary unknowns</u>

The intermediary unknowns have no further use in the description of our figure. Their sole purpose was the calculation of the main unknowns **α** and **R**, used in later equations.
Of the secondary unknowns, we found **c** and **x** useful:
The length of each chord is:

$$\boldsymbol{c = 2R\ \sin\frac{\alpha}{2}}$$

The length of each arc of circle (or side of the curve) is:

$\boldsymbol{x = R\ \alpha}$ (with **α** measured in radians)

And the perimeter: $\boldsymbol{P = n\ x}$

Surface of the figure

Whatever the number of sides of the original polygon, the area has the same expression, being a function of $\boldsymbol{\alpha}$ and **R**. Let $\mathbf{A_R}$ be the area of the figure, A_p the area of the inner polygon formed by the chords and A_s the area of each segment of circle. Then:

$$A_R = A_p + n\,A_s\ (\mathbf{1})$$

with:

$$A_p = \frac{1}{4}nc^2\cot\left(\frac{u}{n-2}\right)\ (\mathbf{2})$$

and:

$$A_s = \frac{1}{2}R^2(\alpha - \sin\alpha)\ (\mathbf{3})$$

substituting **c** in the equivalent, but more appropriate form:

$$c = R\sqrt{2(1-\cos\alpha)}\ (\mathbf{4})$$

We find:

$$\boldsymbol{A_R = \frac{1}{2}nR^2\left[(1-\cos\alpha)\cot\left(\frac{u}{n-2}\right) + \alpha - \sin\alpha\right]}$$

C. Statement and proof of the conjectures

There are four underlying conjectures which were used in our calculations. The third and the fourth were proved by reduction to the first and the second. We gave an approximate proof for the second conjecture and no analytical proof for the first.
The first conjecture, although not true, provides a very good approximation. We offer below a theoretical calculation of the area error as a percentage of the Reuleaux area for the particular case n=4 (square). This is intended to show exactly how small the error is between the real figure and the Reuleaux approximation.
Due to the inherent limitations of the programme the case when N approaches infinity cannot be truly tested. Hence, upon comparing the distance from the center of an arc to the midpoint of that arc with the distance from the same center to one of the vertices found at the ends of the arc, one finds slightly different results. For N=10000 divisions the largest error is found for n=3 (Reuleaux triangle), and its value is about 6.45%, but drops to only 1.35% for n=4. As the number of sides n of the original polygon increases, the error becomes even smaller.

1. Statement of the conjectures

Conjecture 1:

Let n be the number of sides of the original regular n-sided polygon $S_1S_1^{(n-1)'}$ and N the number of divisions of each side. When $N \to \infty$ the figure resulting from the intersection of the nN segments of the type $S_k^{i'}S_k^{j'}$ (where i and j go from 0 to n-1 satisfying either $|j-i| = 1$ or $|j-i| = n-1$ and k goes from 2 to N) is:

a) a Reuleaux n-gon if n is odd
b) an "inflated" n-gon with arcs of circle as sides if n is even

Conjecture 2:

Let $V_{1,n}$ be the vertices of the polygon with sides arcs of circle (denoted by $x_{1,n}$), $C_{1,n}$ the centres of the circles whose intersections generate the curved structure and P_k the middle point of the arc x_k. Then, if:

a) n is odd: $C_k=V_k$ for k in range $[1, n]$
b) n is even: $C_k=P_k$ for k in range $[1, n]$

The size of this figure is given by its radius, denoted by R and calculated as follows:

a) n is odd: $R = d(C_a, P_a)$ where C_a is the center of an arc and P_a is the midpoint of that arc
b) n is even: $R = d(C_a, C_b)$ where C_a and C_b are opposite centers

Conjecture 3:

Let P_k be the midpoint of the arc x_k and $M^{i'}$ the midpoint of the side $S_1^{i'}S_1^{(i+1)'}$. We define the distances $d = d(S_1^{i'}, P_{i+1})$ and $y = d(M^{i'}, V_{i+1})$ for i in range $[0, n-1]$. Then, if:

a) n is odd: $R_{odd} = H_{odd} - d - y$ where H_{odd} is the height of the original polygon
b) n is even: $R_{even} = H_{even} - 2d$ where H_{even} is the longest diagonal in the original polygon

Conjecture 4:

Let $M^{i'}M^{(i+1)'}$ be the middle lines of the polygon $S_1S_1^{(n-1)'}$ and P_i the midpoints of the arcs x_i for i in range $[0, n-1]$. Then, $M^{i'}M^{(i+1)'} \cap x_i = \{ P_i \}$. This means that the middle lines are tangent to their corresponding arcs in their midpoints.

2. Proof of the conjectures

Note: These proofs are schematic and are given in the order in which they were done.

Conjecture 4:

We consider two distinct cases for the number of divisions: N=even and N=odd.

a) N is even: $M^{i'}M^{(i+1)'}= S_{N/2+1}^{i'}\, S_{N/2+1}^{(i+1)'}$ which is the same as saying that the middle lines coincide with the central division segment (1). According to the first conjecture, when $N \to \infty$ the portion of $S_{N/2+1}^{i'}\, S_{N/2+1}^{(i+1)'}$ within the "inflated polygon" is reduced to only one point, more precisely to its midpoint, due to symmetry (2). From relations (1) and (2) the conjecture is proved.
b) N is odd: $S_{N-k}^{i'}\, S_{N-k}^{(i+1)'} \cap S_{N-k+1}^{i'}\, S_{N-k+1}^{(i+1)'} = \{ Q \}$ or the intersection of the two central division segments is Q (1). However, Q is also the midpoint of $M^{i'}M^{(i+1)'}$ (the middle line) because $M^{i'}$ is the midpoint of $S_{N-k}^{i'}\, S_{N-k}^{(i+1)'}$ (2). But, according to the first conjecture, when $N \to \infty$ Q becomes part of an arc of circle, also a side of the "inflated" polygon (3). From the three relations, the conjecture is proved.

Conjecture 3:

We consider two distinct cases for the number of sides: n=even and n=odd.

a) n is even: From C1, C2 and symmetry we find that $S_1^{a'}$, C_{a+1}, C_{b+1} and $S_1^{b'}$ are collinear in this order, where $S_1^{a'}$ and $S_1^{b'}$ are opposite vertices of the original polygon and C_{a+1} and C_{b+1} are the radially opposed centres of the arcs corresponding to $S_1^{a'}$ and $S_1^{b'}$ (1). From C2 we know that $R = d(C_{a+1}, C_{b+1})$ (2). From the definitions of C3 we know that $d = d(S_1^{a'}, C_{a+1}) = (S_1^{b'}, C_{b+1})$ (3). For an even-sided polygon H_{even} is the longest diagonal: $H_{even} = d\left(S_1^{a'}, S_1^{b'}\right)$ (4). From these four relations, we get: $R_{even} = H_{even} - 2d$, thus the conjecture is proved.

b) From C1, C2 and symmetry we find that $S_1^{i'}$, P_{i+1}, C_{i+1} and $M^{(i+2)'}$ are collinear in this order, where $S_1^{i'}$ is a vertex of the original polygon, C_{i+1} the centre of the arc whose outer curvature opposes $S_1^{i'}$, P_{i+1} the midpoint of this arc and $M^{(i+2)'}$ the midpoint of the side which opposes $S_1^{i'}$ (1). From C2 we know that $R = d(C_{i+1}, P_{i+1})$ (2). From the definitions of C3 we know that $d = d(S_1^{i'}, P_{i+1})$ (3) and that $y = d(M^{(i+2')}, C_{i+1})$ (4). For an odd sided polygon H_{odd} is the height of the polygon: $H_{odd} = d\left(S_1^{i'}, M^{(i+2)'}\right)$ (5). From these five relations, we get: $R_{odd} = H_{odd} - d - y$, thus the conjecture is proved.

Conjecture 2:

As before we treat the two distinct cases:

a) n is even: From the first conjecture, we know that the curve is an inflated polygon with the same number of sides as the original polygon (1). As is the case with any even-sided polygon, each side opposes another side (2). From these two statements, we find that each arc of circle opposes another arc of circle. The centre of each arc of circle must thus be, due to symmetry, the midpoint of its opposing arc, and vice versa. This also implies that the radius is the distance between these two midpoints.

b) n is odd: In the Reuleaux polygon, like in any odd-sided polygon, each side (here, arc of circle) opposes a vertex. Therefore, for reasons of symmetry and in order for each side to be an arc of circle, each vertex needs to be the centre of the opposing arc, and thus the radius the distance between the vertex and any point situated on the arc.

D. Discussion on the first conjecture

In order to calculate the area error, we need to find the equation of each side of the curve. Below we derive the parametric equations of a side's envelope for the general case, using skew coordinates (the axes are tilted at the angle u of the original polygon).

D.1. Parametric equations

We begin with a drawing (figure 6) for the particular case $n = 4$ ($u = 90°$) and $N = 8$:

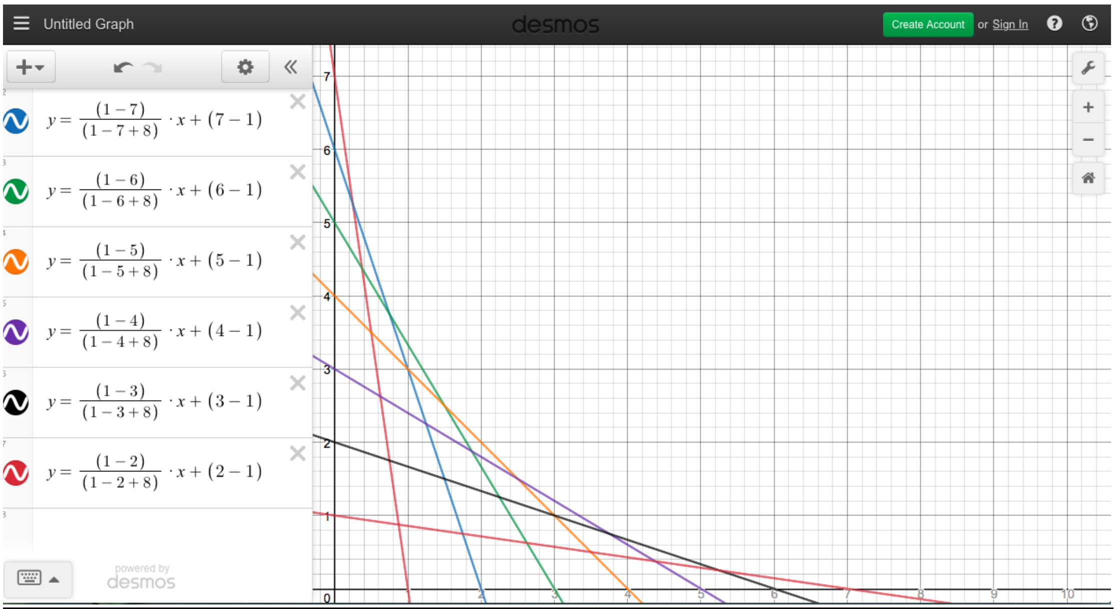


fig.6- 8 division segments in orthogonal coordinates

Generalising the situation above for skew coordinates by introducing a *cos(u)* and a *sin(u)* factors for the abscissa and the ordinate respectively, we found the parametric equation of any segment:

$$(p-1)\boldsymbol{x} + \frac{N-(p-1)(cos(u)+1)}{sin(u)}\boldsymbol{y} - (p-1)[N-(p-1)] = 0$$

where the integer parameter p goes from 1 to N, N being the number of divisions and u the angle between the coordinate axes. This equation will be more simply written as:

$$F(x, y, p) = 0$$

Differentiating the equation with respect to the parameter p, we find:

$$\frac{\partial F(x, y, p)}{\partial p} = 0$$

or, more exactly:

$$x - \frac{cos(u)+1}{sin(u)} y - N + 2(p-1) = 0$$

Solving the system formed by the two equations, we get:

$$\begin{cases} \boldsymbol{x = \dfrac{[N-(p-1)]^2 + (p-1)^2 cos(u)}{N}} \\ \boldsymbol{y = \dfrac{(p-1)^2 sin(u)}{N}} \end{cases}$$

These equations correspond to the envelope of the bottom left side of the curve. The representation of this curve is given in red for the particular case $n = 4$ ($u = 90°$) and $N = 8$ (figure 7):

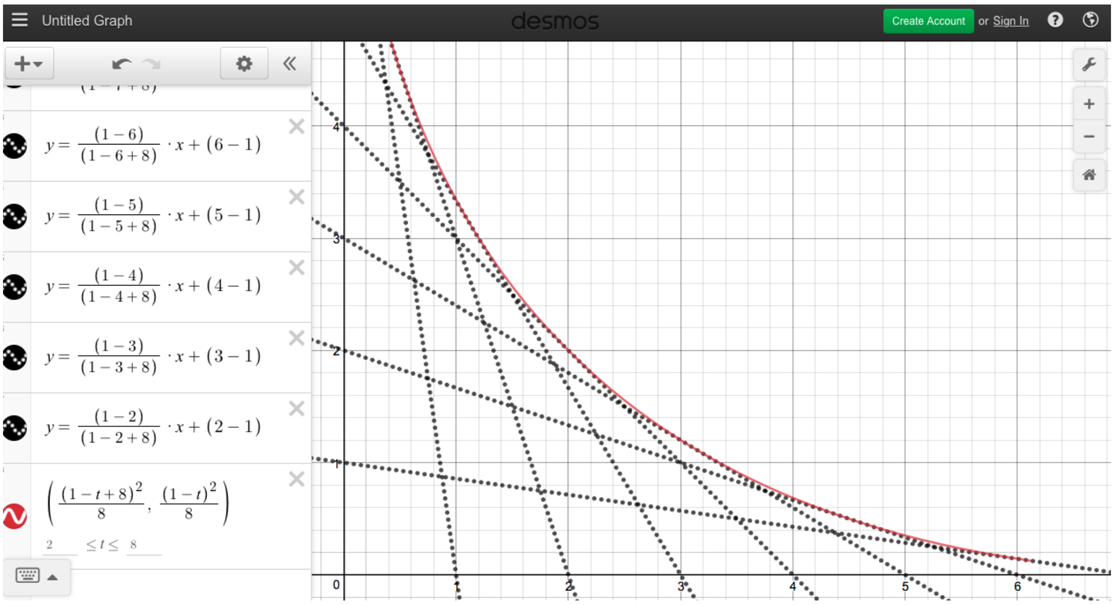


fig.7- The envelope overlaid on figure 6

Due to the *cos(u)* and a *sin(u)* factors, the implicit equation obtained from the parametric equations is rather unusable for the general case. If $u = 90°$, these factors become 0 and 1 repectively. This is why we further analyse the particular case $n = 4$.

D.2. Analysis of the particular case $n = 4$ ($u = 90°$)

We first need to find the equations of 3 adjacent sides, as illustrated in figure 8:

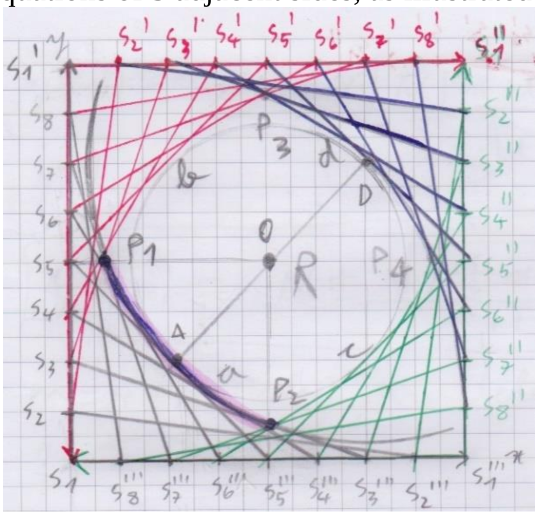


fig.8- All 4 envelopes

Based on the general parametric equations, we find the equations of the bottom left side (curve a):

$$a: \begin{cases} x = \dfrac{[N-(p-1)]^2}{N} \\ y = \dfrac{(p-1)^2}{N} \end{cases}$$

Analogously, the equations of the other 2 adjacent sides are:

$$b: \begin{cases} x = \dfrac{(p-1)^2}{N} \\ y = \dfrac{(p-1)[2N-(p-1)]}{N} \end{cases}$$

$$c: \begin{cases} x = \dfrac{N^2-(p-1)^2}{N} \\ y = \dfrac{[N-(p-1)]^2}{N} \end{cases}$$

Converting these equations in implicit form, we find:

$$a:\ y = \left(\sqrt{x}-\sqrt{N}\right)^2$$

$$b: y = 2\sqrt{Nx} - x$$

$$c: y = 2(N - \sqrt{N(N-x)}\,) - x$$

We now need to find the intersection points P_1 and P_2:

$$\{P_1\} = a \cap b$$

$$\{P_2\} = a \cap c$$

which give:

$$\{P_1\}: \begin{cases} x_1 = \dfrac{\left(3-2\sqrt{2}\right)N}{2} \\ y_1 = \dfrac{N}{2} \end{cases}$$

$$\{P_2\}: \begin{cases} x_2 = \dfrac{N}{2} \\ y_2 = \dfrac{\left(3-2\sqrt{2}\right)N}{2} \end{cases}$$

We now need to find the distance $R = [AD]$, where A and D are the midpoints of the opposite curves a and d.

The conditions for the point A are:

$$\begin{cases} x_A = y_A \\ A \in a \Leftrightarrow y_A = \left(\sqrt{x_A}-\sqrt{N}\right)^2 \end{cases}$$

which give:

$$\{A\}: \begin{cases} x_A = \dfrac{N}{4} \\ y_A = \dfrac{N}{4} \end{cases}$$

Let O be the centre of the figure. Thus:

$$\{O\}: \begin{cases} x_O = \frac{N}{2} \\ y_O = \frac{N}{2} \end{cases}$$

By symmetry with respect to O, it follows that:

$$\{D\}: \begin{cases} x_D = \frac{3N}{4} \\ y_D = \frac{3N}{4} \end{cases}$$

Therefore, the radius R is:

$$R = [AD] = \sqrt{(x_D - x_A)^2 + (y_D - y_A)^2} = \frac{N\sqrt{2}}{2}$$

Let S_R denote the area under the arc of circle on the interval $[x_1, x_2]$ and S_a the area under the curve a on the same interval. A_R is the total area of the curve, as given by the arc of circle approximation. Figure 9 shows the small diference between S_R and S_a:

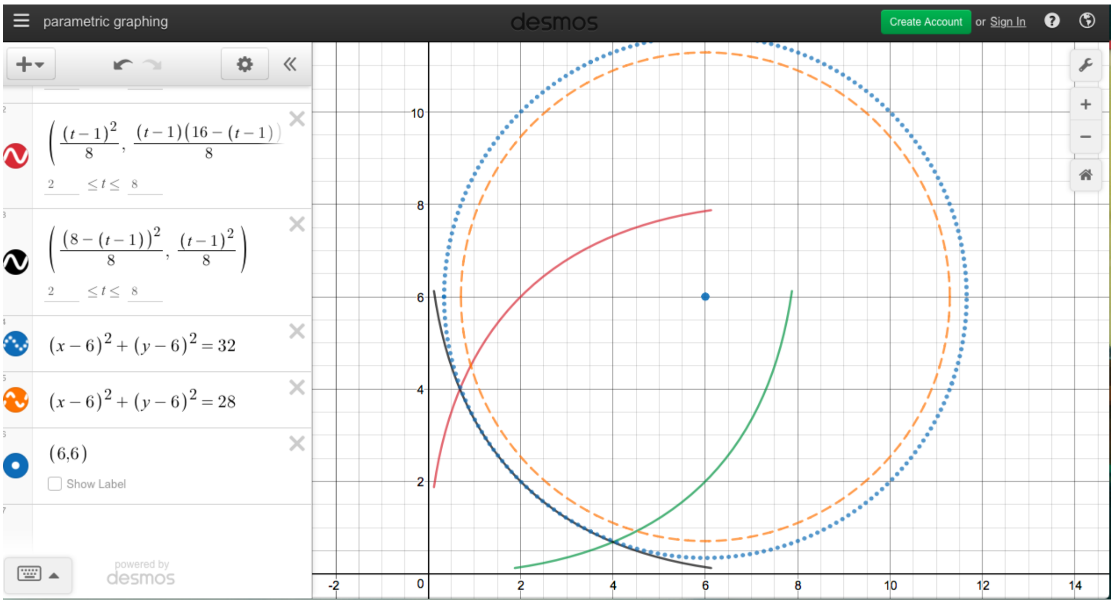


fig.9- Representation of the curve and of the circle used as approximation

This drawing shows that the curve a is not an arc of circle. Detailing on the interval $[x_1, x_2]$, we see the difference even better (curve a in blue, arc in red):

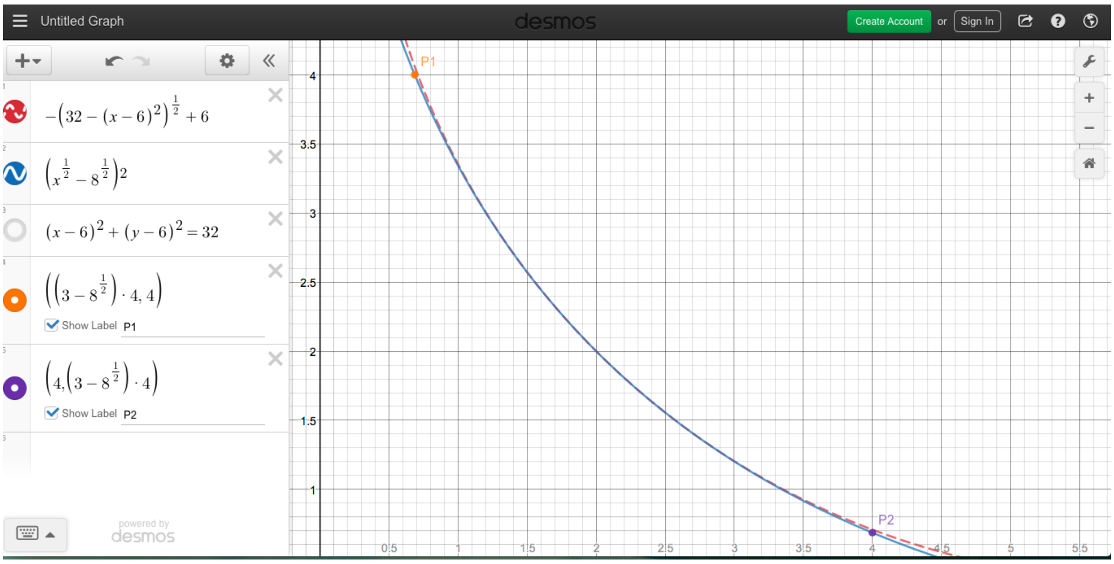


fig.10- Detail on figure 9

We notice that $S_R > S_a$. Thus, the total area error is:

$$\Delta S = -\frac{4(S_R - S_a)}{A_R}$$

Until here all coordinates were measured on a scale from 0 to N. Considering the side of the square to be l, then it is enough to replace N with l in all equations. Hence, the equation of the curve a becomes:

$$y_a = \left(\sqrt{x} - \sqrt{l}\right)^2$$

and the equation of the arc of radius R centred in D:

$$(x - x_D)^2 + (y - y_D)^2 = R^2 \Longrightarrow$$
$$|y - y_D| = \sqrt{R^2 - (x - x_D)^2}$$

But as we study the lower half of the circle, we find after replacing the coordinates of D:

$$y_R = \frac{3l}{4} - \sqrt{\frac{l^2}{2} - \left(x - \frac{3l}{4}\right)^2}$$

Therefore, we get:

$$S_a = \int_{x_1}^{x_2} y_a \, dx = \int_{\frac{(3-2\sqrt{2})N}{2}}^{\frac{N}{2}} \left(\sqrt{x} - \sqrt{l}\right)^2 dx$$

$$S_R = \int_{x_1}^{x_2} y_R \, dx = \int_{\frac{(3-2\sqrt{2})N}{2}}^{\frac{N}{2}} \frac{3l}{4} - \sqrt{\frac{l^2}{2} - \left(x - \frac{3l}{4}\right)^2} \, dx$$

Calculating the integrals and A_R and symplifying by l^2, we find:

$$\Delta S \cong 1.35\%$$

<u>E. Software implementations</u>

As a means to observe and measure the properties of the geometrical figure formed inside the polygon, we have developed an algorithm for drawing any polygon with its sides divided into a selected amount of equal segments. The code also implements the measurement of any length and angle in the drawing. In order to compare our figure to a Reuleaux polygon, all the mathematical details of both the obtained figure and a similar Reuleaux polygon are calculated and displayed.

To generate the figure, the first step must be drawing the polygon.

```
// draw polygon sides path from corner to corner
for ( var i = 1; i <= numberOfSides; i += 1 ) {

   // define position of corner
   var x = Xcenter + polygonRadius * Math.cos( i * 2 * Math.PI / numberOfSides ),
      y = Ycenter + polygonRadius * Math.sin( i * 2 * Math.PI / numberOfSides );

   // draw line to corner;
   canvasContext.lineTo( x, y );

   // insert new point with position in corners array
   corners.push( {
      x: x,
      y: y
   } );

}
```

Then we generate the divisions of each side in order. The algorithm calculates the positions for the n^th^ points of each segment and then draws a line between them.

```
// define division variables (for storing array and point positions)
var divisionPosX, divisionPosY, sectionPoints = [];

// polygon side division path drawing
// i = corner id
for ( var j = 1; j < numberOfDivs; j += 1 ) {

   // j = current division (first, second, ...)
   for ( var i = 1; i <= numberOfSides; i += 1 ) {

      // define divison point position
      if ( i < numberOfSides ) {
         divisionPosY = corners[ i - 1 ].y + ( corners[ i ].y - corners[ i - 1 ].y ) * j / numberOfDivs;
         divisionPosX = corners[ i - 1 ].x + ( corners[ i ].x - corners[ i - 1 ].x ) * j / numberOfDivs;
      }
      if ( i == numberOfSides ) {
         divisionPosY = corners[ i - 1 ].y + ( corners[ 0 ].y - corners[ i - 1 ].y ) * j / numberOfDivs;
         divisionPosX = corners[ i - 1 ].x + ( corners[ 0 ].x - corners[ i - 1 ].x ) * j / numberOfDivs;
      }
```

```
        // push point into array
        sectionPoints.push( {
            x: divisionPosX,
            y: divisionPosY
        } );

    }

    // draw division path based on stored positions
    canvasContext.moveTo( sectionPoints[ 0 ].x, sectionPoints[ 0 ].y );
    for ( var i = 1; i <= numberOfSides; i += 1 ) {
        if ( i < numberOfSides ) {
            canvasContext.lineTo( sectionPoints[ i ].x, sectionPoints[ i ].y );
        }
        if ( i == numberOfSides ) {
            canvasContext.lineTo( sectionPoints[ 0 ].x, sectionPoints[ 0 ].y );
        }
    }

    // empty array for new set of points
    sectionPoints = [];

}
```

We chose to calculate the surface area of the figure as a percentage of the initial polygon's area. To do this we counted the number of white and black pixels (the lines are drawn with black, the initial polygon has its interior coloured in white and the background is orange). Then we just divided the number of white pixels to the total number of black & white pixels and we found the proportion of the figures area which we could then transform into a percentage to be displayed on screen.

```
// get data of all pixelArrays in canvas
var imageData = canvasContext.getImageData( 0, 0, canvas.width, canvas.height );
var pixelArray = imageData.data;
var numberOfPixels = pixelArray.length;

// THEORY
// each pixel has a red, green, blue and alpha(transparency) property
// these properties are stored as elements of the array in this order
// red had id 0, green 1, blue 2, aplha 3

// EXAMPLE
// to acces the blue(id = 2) value of pixel 367 we will get pixelArray(367 + 2)

// defining color value id's
var r = 0,
   g = 1,
   b = 2,
   a = 3;
```

```
// iterate through each pixel
for ( var p = 0; p < numberOfPixels; p += 4 ) {

   // verify if pixel is black or white (no racism intended)
   if ( pixelArray[ p + r ] == 255 && pixelArray[ p + g ] == 255 && pixelArray[ p + b ] == 255 ) {
      white++;
   }
   if ( pixelArray[ p + r ] == 0 && pixelArray[ p + g ] == 0 && pixelArray[ p + b ] == 0 ) {
      black++;
   }

}

// For large values for the number of divisions we will obtain
// a white Reuleaux Polygon in the black Parent Polygon.
// By calculating the percentage of the white area from the initial black polygon
// We can verify if our formula for the Reuleaux Polygon Area is correct
var experimentalPercentage = ( 100 * white / ( white + black ) ) + "%";
```

We also implemented algorithms for measuring the distance and angle between consecutive clicked points on the drawing or for calculating the approximate values of the figure's properties by supposing it is a Reuleaux polygon, as the error is only around 1% between the properties of the two (for n greater than 4).

The evolution of the error, as predicted by the programme is given by the chart below:

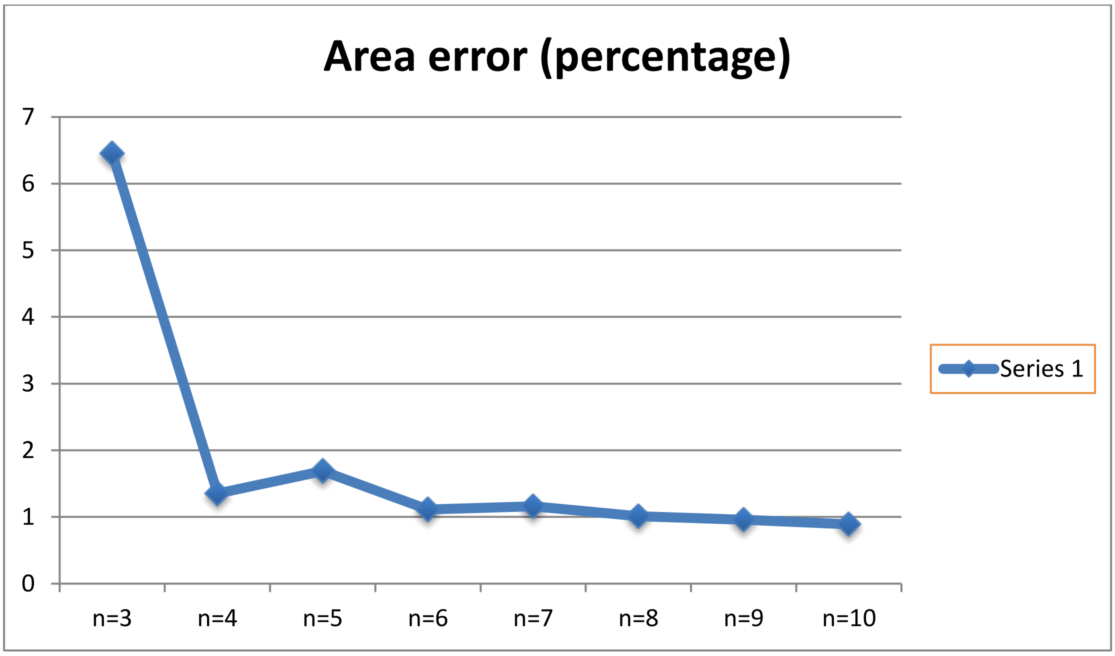


Fig.11- The evolution of the error as the number of sides increases

You can read the full code here:

https://github.com/Rototu/matlan/blob/gh-pages/main.js

F. Applications in real life

"Inflated" polygons are known since the Middle Ages, when they were used as decorative motives, such as in church windows, especially the Reuleaux triangle, which incorporated both the triangle of Trinity and the perfection of the circle:

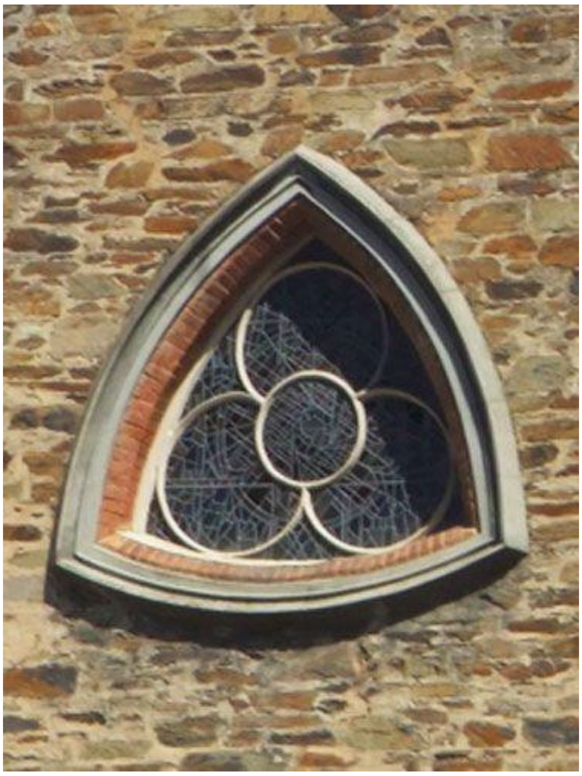


fig.12- Reuleaux-shaped window

Also, due to the fact that Reuleaux polygons are isodiametric curves or curves of constant width - whose width (defined as the perpendicular distance between two distinct parallel lines each having at least one point in common with the shape's boundary but none with the shape's interior) is the same regardless of the orientation (only the curves with n odd) - they have a variety of applications, including:

1. Coins: due to the constant width, they can rotate freely inside detectors (from left to right: 5 rupees, 50 pence, 1 Australian dollar)

fig.13- 5 rupees

fig.14- Fifty pence

fig.15- 1 AUD

2. Engine components, such as the central "cylinder" of the rotary engine found on cars such as the Mazda Rx-8 or Mazda Furai:

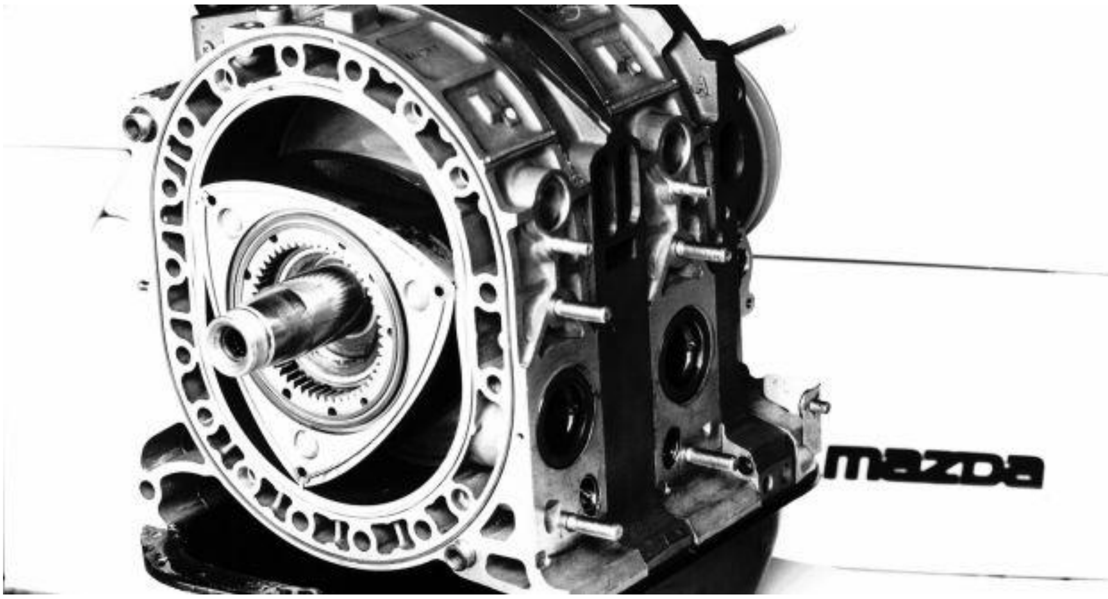


fig.16- Mazda rotary engine

3. Water valve cover: why not?

fig.17- Water valve cover in San Francisco

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This article is written by students. It may include omissions and imperfections.

Avalanche Simulator

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Presentation of the research topic

Let's consider a *n x n* grid with an integer number of snow fakes. If in a moment a cell contains at least 4 fakes, it is unstable: it avalanches and it's giving all its flakes to the 4 neighbouring cells. If the avalanching cell is on the border, the flakes that would fall outside the grid are considered lost.

Brief presentation of the conjectures and results obtained

We analysed the evolution of the system according to the imposed rules with reference to the proposed problem, starting from some restrictive hypothesis. We put all *N* flakes in the central cell of an unlimited grid *n x n*, without counting any lost flake. Flakes propagate until reaching stability after a number of steps, occupying a certain number of cells that let determinate a maximum radius *r* that is the maximum number of cells neighbouring to the central one. In this work, we will analyse the relation between the number *N* of flakes in the starting cell and the radius *r* of the stability's configuration.

Contents:

<u>Formulation of the problem</u>

We put in each cell of a grid of *n x n* cells an integer number *N* of snowflakes. Let this be the starting configuration. The model evolves as follows: if a cell *z(x, y)* where *x* and *y* are respectively line *x* and column *y*, contains at least 4 flakes, it's called unstable and loses 4 flakes which switch to the four neighbouring cells, according to the rule:

$$z(x,y) \to z(x,y) - 4;$$
$$z(x \pm 1, y) \to z(x \pm 1, y) + 1;$$
$$z(x, y \pm 1) \to z(x, y \pm 1) + 1.$$

If this cell is on the border, the flakes which should be given to the neighbouring cells are lost. In this way, every unstable cell determines the evolution of the grid's state.

Restrictive hypothesis

We fixed some restrictive hypothesis to have an easier approach to the problem:
1. *N* flakes were placed only in the central cell of the grid *n x n*;
2. no restriction wasn't put to the grid's dimensions, so for the moment the condition of loss of flakes was excluded.

Simulation tools

We used a numerical approach to simulate the avalanche, building appropriate software that, given the starting situation (insertion of the *N* flakes in the central cell), simulate the evolution of the system until reaching the condition of stability, where no cell contains a number of flakes $z > 3$. Basically, the algorithmic approaches consist in two nested loops that produce the evolution in every cell.

Commutativity of the algorithm

One of our first questions was the commutativity of the algorithm, that is we asked ourselves if the final state of the matrix for every step depends or not on the order of its journey. We noticed that, checking line by line or column by column, the results are the same with all the built tools.

Spreadsheet

First, we used an Excel spreadsheet to simulate the evolution of the system after putting *N* flakes in the central cell. The evolution was simulated with two different algorithms:
1. using the library functions of the program, building operations between cells using conditional instructions;
2. building a macro in VB script which automatically gives the configuration of the next step specifying when the system is stable. The two proceedings give the same results. The second one is more advantageous because it allows a check of every step.

C++ algorithm

The C++ algorithm expects the initialization and the upload of the data relating to the configuration in the form of a matrix with side *n*. Then the program individually checks every cell and, when it finds a value greater than or equal to 4, it applies the following algorithm:

```
pos[i][j]-=soglia;

pos[i][j-1]=++;

pos[i][j+1]=++;

pos[i-1][j]=++;

pos[i+1][j]=++;
```

It goes on like this for each cell of the matrix until it completes a full step. Then it searches the maximum value z in each cell of the matrix and compares it with the limit ($z_{max} = 4$). If $z \geq z_{max}$, it checks again the matrix applying the same algorithm for every unstable cell. The program continues until the maximum value, that is searched at the end of every step of the matrix, is less than the limit in such a way that all the cells are stable.

Java algorithm

This java application's main purpose is the graphical representation of the propagation of the number N of flakes in the central cell through a colour printing. Unlike the C++ algorithm, this program always saves the position of unstable cells, and then it works only in those, just like the algorithm in C++. Below, the main part of the algorithm is reported:

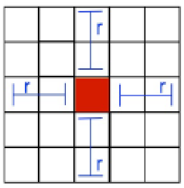
```
List<int[]> rowList = new ArrayList<int[]>();
for(int i=1;i<=bounds;i++){
for(int j=1;j<=bounds;j++){
if(mat[i][j]>=4)
rowList.add(new int[] {i,j});
}
}
for (int[] is : rowList) {
crolla(is,mat);
}
```

<u>Data analysis</u>

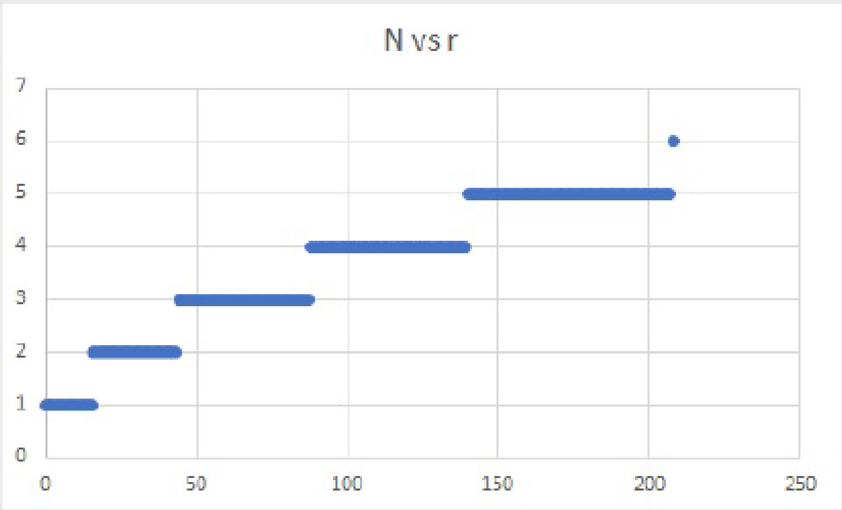
Analysed variables

Starting from an initial configuration with N flakes in the central cell, we made the system evolve until reaching a stability situation, so that no cell contains a number of flakes z > 3. In this situation, we determined the radius r defined as greatest distance in cells from the central point. As said before, we chose to study the evolution of the avalanche defining two parameters:
- the number of flakes initially contained in the central cell N;
- the radius r, defined as the greatest distance in cells from the central one.

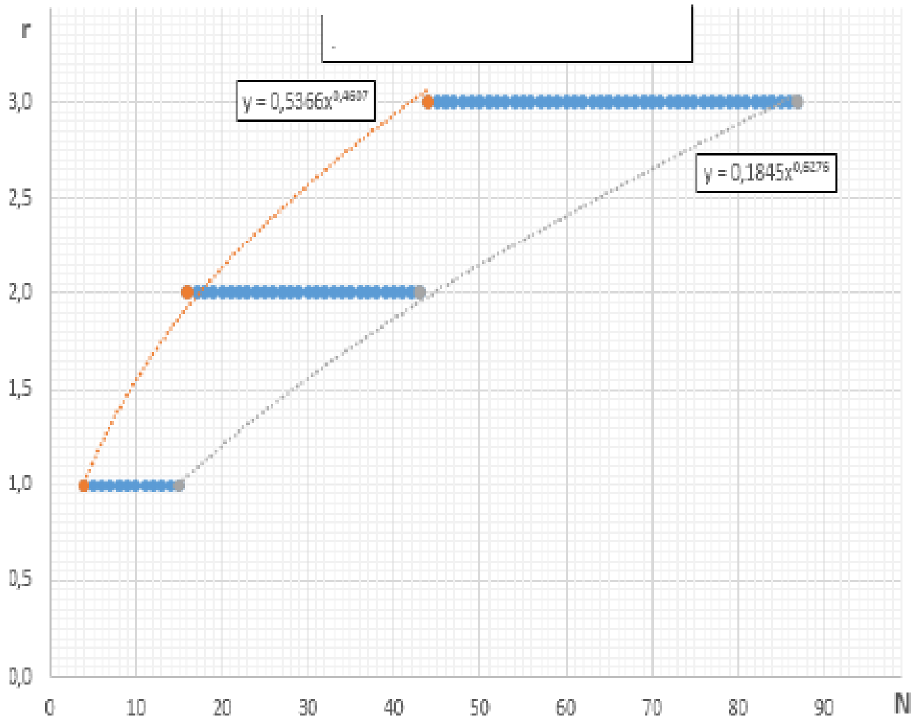
Beginning from a starting configuration with N flakes in the central cell, we made the system evolve until the equilibrium, where no cell contains a number of flakes z > 3. In this situation, we determined the radius r.



All procedures we used to simulate the avalanche unequivocally show that a direct relation between the variables N and r subsists, meaning that, as we always expect, increasing the number N of flakes in the central cell, the radius r tends to increase, too. We simulated the evolution of the system with our instruments until N = 20000.

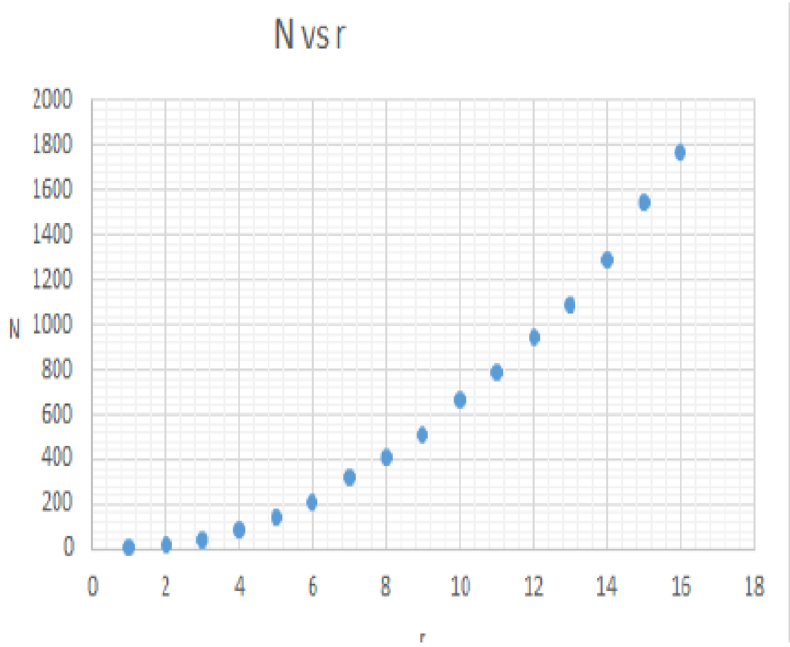


The graph above shows the intervals of *N* with constant radius *r*.
It is right to hypothesize the presence of a certain functional relation between the two variables like *r = f(N)*, that will be a step function. So, we decided to represent the relation between the number *N* of the start of every step and the corresponding radius *r*.



The graph represents two curves that pass through two points *P(N, r)*, where *N* indicates respectively the starting and the ending of the step, defined as the range where the radius *r* is constant.

If you examine the inverse relation between N and r you can notice the presence of a quadratic relation like $N \cong r^2$.

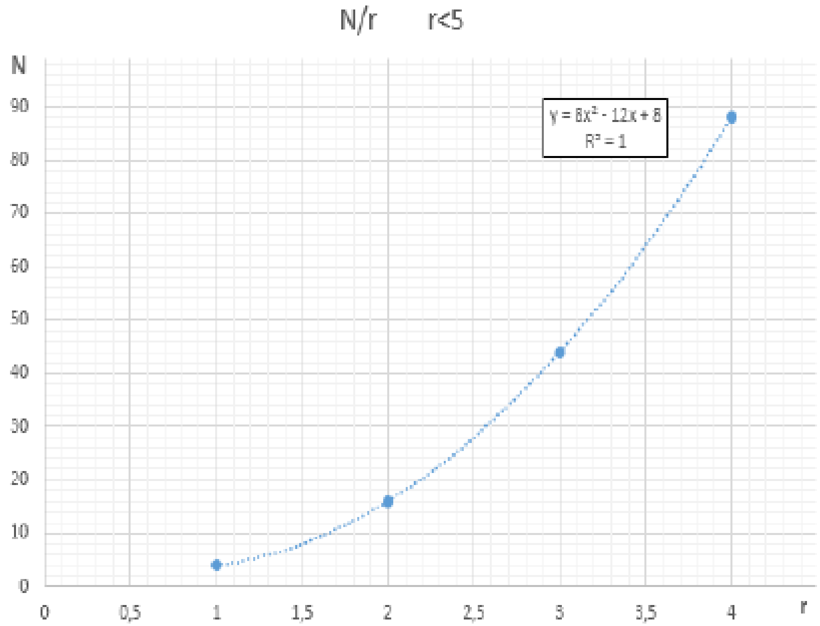


We fitted experimental data in a polynomial model of second degree using the automatic procedure of Excel.
However, we got an exact fit on the points relating to r < 5 but not on the ones greater than *r*.

$r \leq 4$ fit
We realized a graph only with the first 4 couples of values *r* and *N*. If you have a look to the graph below, you can see that a parabola of equation:

$$N = 8r^2 - 12r + 8$$

passes through all the points *P(r; N)*, starting from *r = 1* and *N = 4* to *r = 4* and *N = 88*.



Note the integer values of the coefficients.
As we will discuss below, this law is not compatible with high values of r.

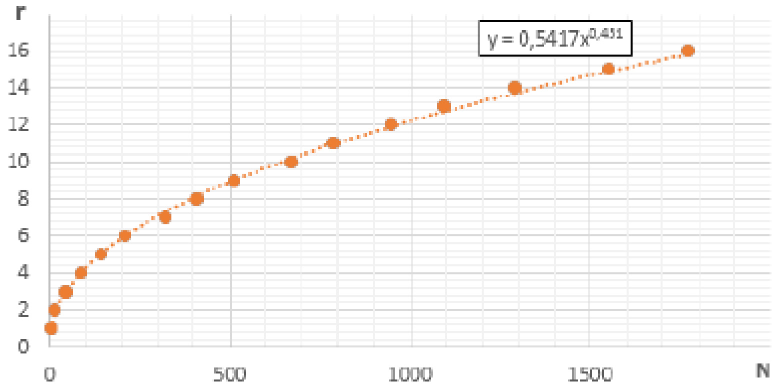
r > 4 **fit**

When the number of flakes N is greater than 144 and then r is greater than or equal to 5, the quadratic function (parabola) stops fitting exactly. Indeed, coefficients resulting from the fit are no longer integer numbers and have consistent variations when you add a new point.
So, it is necessary to represent the trend of the points with a different law. Following the example [1], we tried to search a relation that was a power law like:

$$r(N) = a \cdot N^b$$

However, we need to study the relationship *r(N)* and not *N(r)*.
We continued to determine the couple of parameters a and b through the spreadsheet's automatic function, which allows a better approximation of experimental data for all the 16 couples of available values N and r, finding out:

a = 0,5417 b = 0,4510



In the graphic, you can see that the power law constitutes an acceptable approximation of the trend of experimental points.

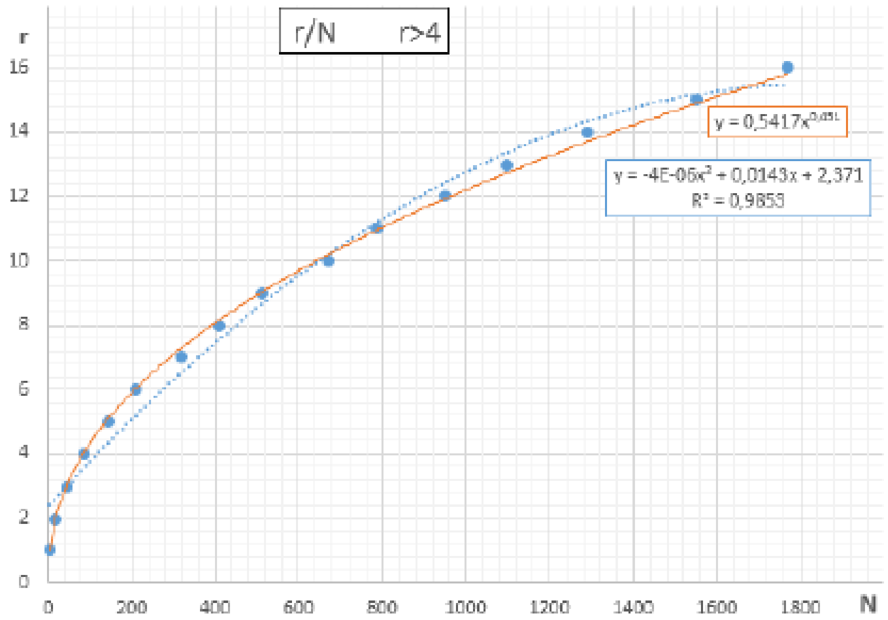
Comparison between the two models

We compared the two models: parabola and power law. For the approximation with a polynomial function of second degree we had to reverse the relationship
$N(r) = ar^2 + br + c$ writing it as:

$$r = \frac{-b+\sqrt{b^2-4a(c-N)}}{2a} \qquad (1)$$

In order to get a criterion to establish which one of the two models (parabola and power law) best describe the phenomenon, we considered 4 couples of points, then we added a couple of points every time until *r = 16*, we calculated the fit's parameters and the squared deviation relative to the relations given by Excel graphs and defined as the difference between the observed value and the calculated one. Calculating the sum of the squared deviations, given that the sum of the squared deviations is equal to 0, unlike what happens in the relation given by the power law, we can notice that the parabola is more precise than the power law if we consider the first 4 points. Increasing the number of couples of points, the sum of the squared deviations calculated on the relation given by the parabola is no longer equal to 0. Even though the relation given by the power law is not precise, the sum of squared deviations is anyway less than the one given by the parabola.

Below is reported a graph representing the two curves.



We can notice from the graph that the power law's curve is much more precise than the parabola's curve, allowing a better approximation of the experimental data.
The power law is indicated by the following equation:

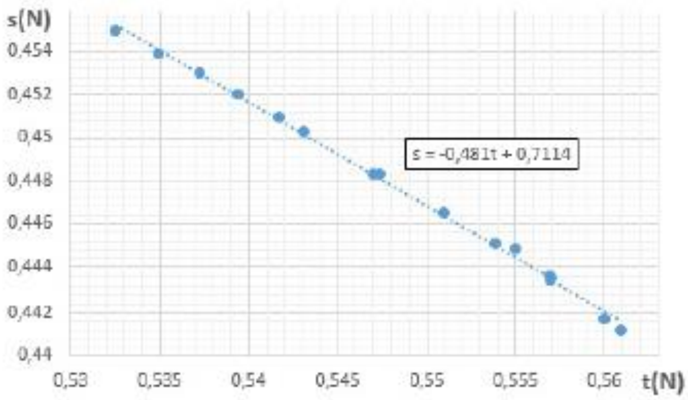
$$y = 0{,}5417x^{0{,}451}$$

As we already noticed, increasing the number of the points used in the fit, we often found consistent variations of the *a* and *b* parameters of the polynomial model, whereas the parameters of the power law model have more similar values, even though they present variations.

r	N	$a(N)$	$b(N)$
5	144	0,5569	0,4436
6	208	0,5550	0,4449
7	320	0,5609	0,4412
8	408	0,5600	0,4417
9	512	0,5570	0,4435
10	672	0,5569	0,4435
11	788	0,5538	0,4451
12	948	0,5510	0,4465
13	1096	0,5470	0,4483
14	1288	0,5473	0,4483
15	1552	0,5431	0,4503
16	1768	0,5417	0,451
17	1960	0,5394	0,452
18	2208	0,5372	0,453
19	2456	0,5349	0,4539
20	2708	0,5325	0,455

Table 1: Coefficient $a(N)$ and exponent $b(N)$ obtained from the power law graphic for every r and N, $r = a \cdot N^b$

Observing the data reported in the table [1], we wonder if a relation between *a(N)* and *b(N)* could exist.
Using 20 couples of values of *a(N)* and *b(N)*, we searched for a fit between these two parameters noting the presence of a linear relation.

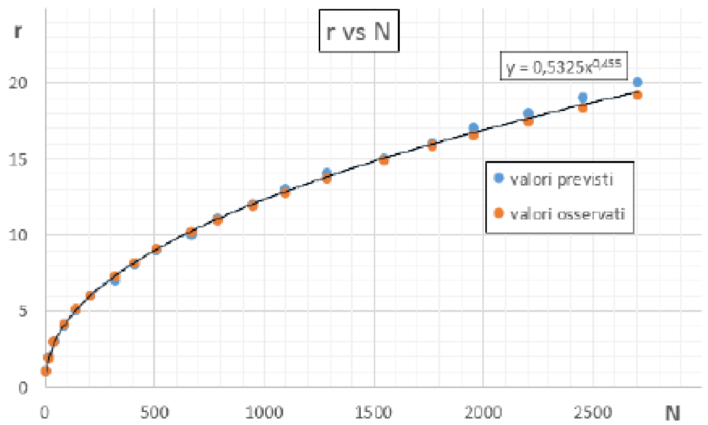


The relation that emerges is:

$$b = -0{,}481a + 0{,}7114 \qquad (2)$$

From this, we understand that the power law that approximates the trend of the data depends on a single parameter. If we choose the exponent *b(N)* as the only parameter, the relation of the best fit is:

$$r = \left(\frac{0{,}7114-b}{0{,}481}\right) \cdot N^b \qquad (3)$$



In this graph, we reported the theoretic trends calculated with the law (3) and the experimental data.
Fitting the relation (3), the model that better fits the data is

$$r = 0{,}5325N^{0{,}455}$$

This trend is in accordance with [1.] where the exponent tends to $\frac{1}{2}$ for a high *N*. We also noticed that for a small *N* the model could be approximated by the relation (1): in the limit of a high *N* we find a relation like:

$$r \cong N^{\frac{1}{2}}.$$

So, we can understand that if we added a consistent number of points, the value of *b* of best fit would tend to $\frac{1}{2}$. According to the relation (2), if $b = \frac{1}{2}$, then $a \cong 0{,}44$.

Interpretation of the model

To understand the relation that we found between N and r we can proceed as follows.

In the starting situation, we put N flakes in the central cell of the grid $n \times n$. According to the criteria that we defined for the evolution of the system, N flakes are distributed on the entire structure until reaching the stability configuration without losing any of them. Then we can think that the number of all the flakes stays constant: only its spatial distribution will change, from a starting situation where the N flakes occupy the volume of the central column, to the final situation, where a certain volume corresponding to a certain number of columns is occupied.

We define the occupied volume V related to a certain cell as the volume of a squared-based parallelepiped with unitary side and height h equal to the number of flakes in the cell.

Obviously, the stability situation imposes that:

$$0 \leq h \leq 3.$$

In this way, the starting volume of the central column is:

$$V_0 = N.$$

After the evolution of the system, the final configuration, after a certain number of step t, will have a certain number of occupied cells that, as we have seen, have a circular symmetry around the central cell. The higher the starting number of flakes N, the better this symmetry is.

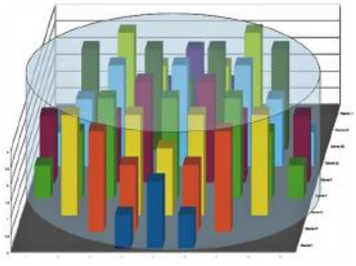
In the graph below the final configuration obtained putting $N = 1 \cdot 10^6$ starting flakes is registered.

As you can see, the distribution has an evident circular symmetry. Different shades of grey are used to represent the occupation values of cells in the final state with the darkest corresponding to 3 flakes and white for 0. The radius of the figure is equal to the matrix radius.

The total volume occupied by the flakes will be constant because the total number of flakes N must be preserved.

The distribution of the occupied cells around the central one with respect to the number N of flakes is very complex, but still allows to define the parameter r. So, as stated before, we can try to make an estimate of the total volume occupied by the flakes.

To do so, we approximate the final cells configuration with a cylinder of radius r and height h.



The cylinder's volume is:

$$V = \pi r^2 \cdot h$$

Considering the volume *V* equal to *N*, we obtain that:

$$r = \frac{\sqrt{\pi h N}}{\pi h} = a \cdot N^{\frac{1}{2}}.$$

This is consistent with the experimental model we found, where $a' \cong 0{,}4$. In fact, we note that for this value of *a* we obtain:

$$0{,}4 = \frac{\sqrt{\pi h}}{\pi h} => h \cong 2.$$

This means that on average the height of every column included on the cylinder of radius *r* is high h = 2, so this means that the average number $\bar{z}$ of flakes in every cell on the equilibrium configuration is equal to 2.
We expect that the average value is $\bar{z} = 1{,}5$, considering that the number of flakes for every balanced cell is $0 \leq z \leq 3$. However, our simulation shows that for high values of *N* we would expect to find a low number of cells with z = 0 of flakes according to the total number of cells included in the basic circumference of the cylinder of radius *r*; this could lead to rise the average number *z* according to 1,5, leading it to a $\bar{z} \cong 2 = h$.

Conclusions

In addressing the avalanches problem, we tried to find a mathematical model that best approximated the relation between *r*, the radius of the stability configuration and *N*, the number of flakes in the central cell.
We found that for a high *N* the relation can be approximated with a power low like:

$$r = a \cdot N^b.$$

According to our simulations, the parameter *b* tends to the value b = 0,5, in agreement with [1.], while the parameter *a* tends to the value a = 0,44, in agreement with the fact that the average number *h* of flakes in each cell in the final stability configuration of the avalanche is near the value $h \cong 2$, noting that the whole configuration assume a circular symmetry.
We can’t provide a rigorous proof of the result nevertheless were confident of it!

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This article is written by students. It may include omissions and imperfections.

Being as Far as Possible

2016- 2017

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The research topic

The topic tackled by our team during October 2016-April 2017 is called: “Being as far as possible”.
Being tackled by both the team from Romania and the one from France, our research has two different interpretations, starting from the same task: “You are situated in a rectangular room with three of your enemies. Where do you have to place yourself in order to be as far from them as possible?
There was one more mention made which informed us of the fact that the distance between us and the enemies actually represents the sum of the distances from each of the three enemies to us.

We tackled the topic by dividing it into two major cases:

1. The enemies are collinear (situated in the same line)
2. The enemies are NOT collinear which means that their positioning forms a triangle.

We began by elaborating the first case:
<u>The enemies are collinear</u>
Firstly, we divided the rectangular room in four equal smaller ones. (see figure 1)

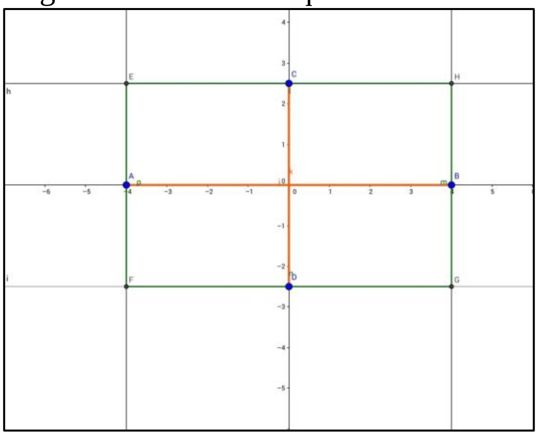

Figure 1: the rectangular divided in four smaller ones

We placed the line on which the enemies were in different parts of the room.
Firstly, the line is perpendicular on the width at ¼ of its length. (fig.2)

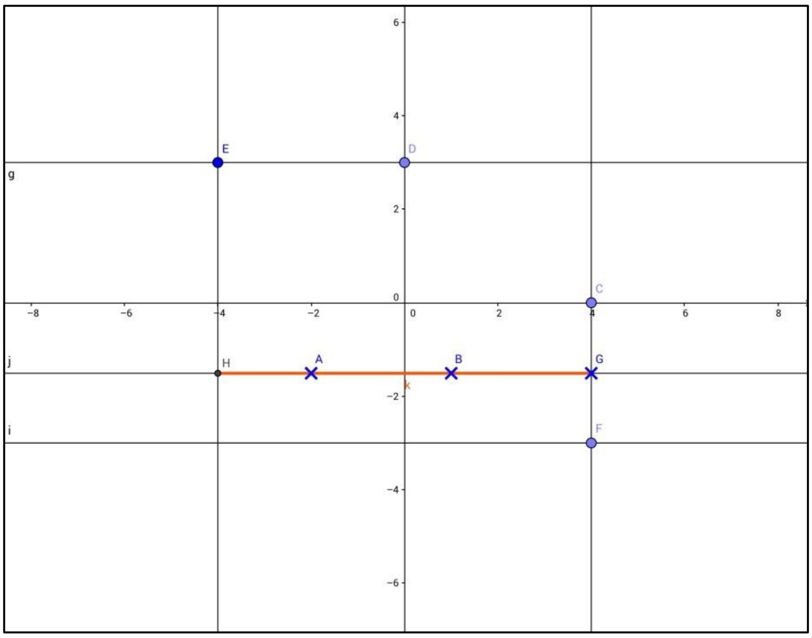

Fig.2

In this case, it means that we have to place ourselves in the furthest part of the rectangular room formed by a parallel of the length, in the corner opposed to the half formed by a line parallel with the length. The corner is going to be the one in the half where only ONE enemy is situated. The half where the majority (two enemies) are, is going to be closer to us, so we are not going to choose that corner for the best positioning.
If the line is situated on the half of the width (fig.3), we only have to take the positioning of the majority of the points reported to the four corners in order to be able to find where we would be the furthest from them. The rest of the procedure is the same as in the previous case.

We also researched the positioning of the enemies on a line which is, this time, vertical, meaning that it is perpendicular on the length and parallel with the width, in different points. For the measurement of the quarter (1/4) (fig.3), as well as for the measurement of the half (1/2) (fig.4), we will be situating ourselves in the furthest half of the room, parallel with the width in the corner opposed to the half where two of the points are.

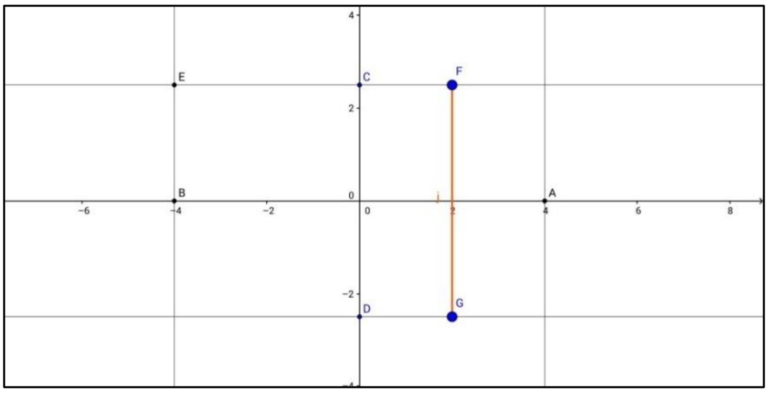

Fig.3

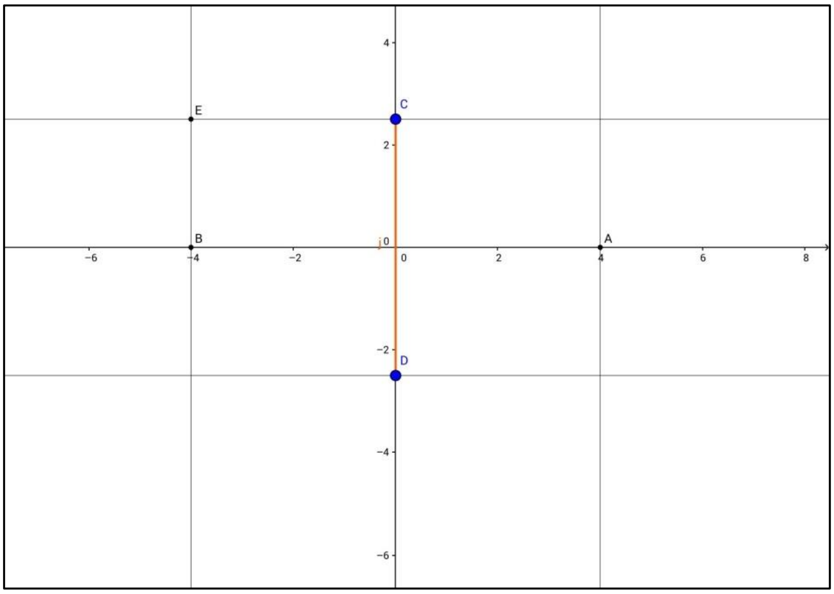

Fig.4

The idea that came to us in order to demonstrate our theory was two develop a formula which calculates the distances between the points, the lengths of different lines which we needed.
It demonstrates that being in any other point of the room would mean that the distance between us and the enemies is smaller than the chosen one (fig.5).

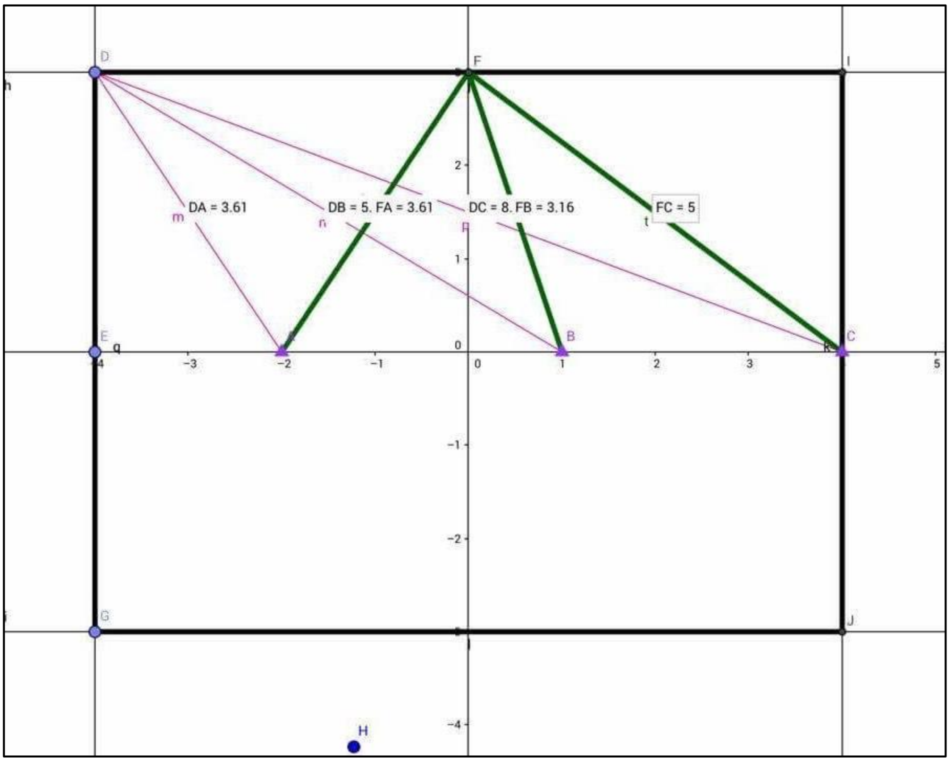

Fig.5

The second case:

Our enemies form a triangle (fig.6)

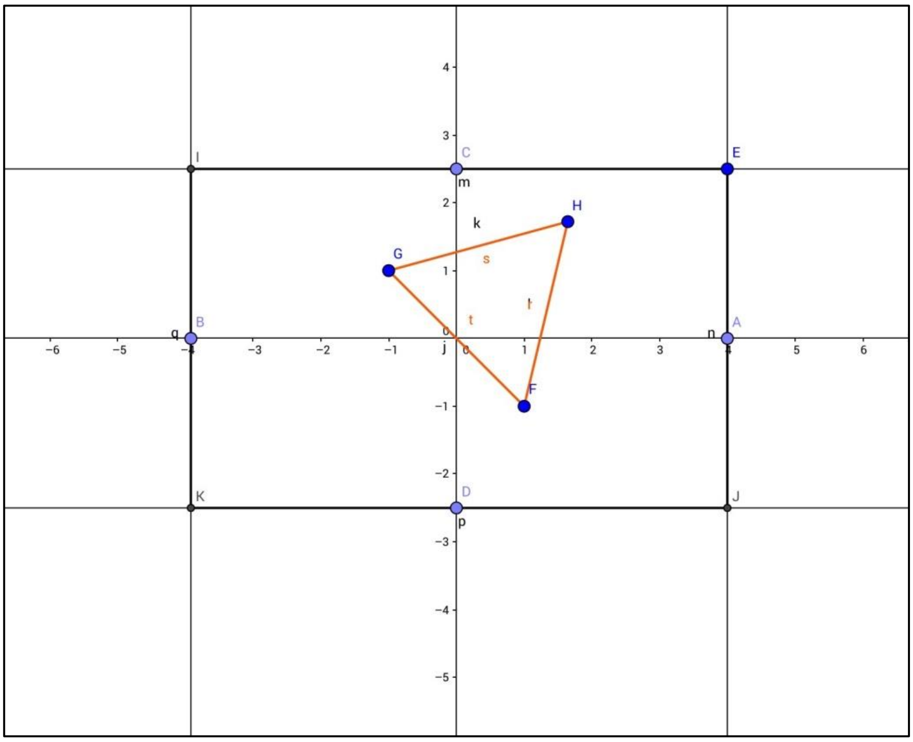

Fig.6

In the particular case in which each of them is in one corner of the room, our best situating will be in the fourth corner of the rectangle. (In this case the points form a right-angled triangle) (fig.7).

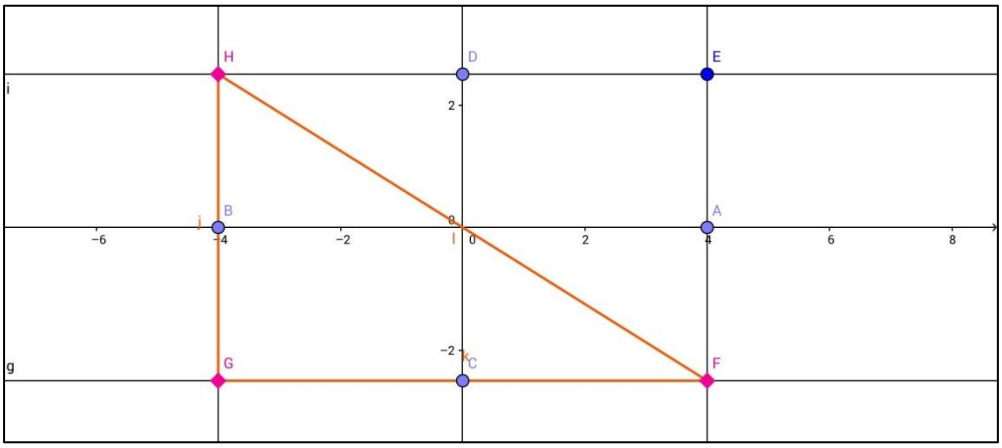

Fig.7

The elaborated formulas are the following ones (we also represented all the mentioned situations in a system helping us to determine the coordinates):

MB=$\sqrt{x_{max}^2 + y_{max}^2 + x_2^2 - 2x_2\sqrt{2(x_{max}^2 + y_{max}^2)} * \cos(MOB)}$

MC= $\sqrt{x_{max}^2 + y_{max}^2 + x_3^2 - 2x_3\sqrt{2(x_{max}^2 + y_{max}^2)} * \cos(MOC)}$

MA=$\sqrt{x_{max}^2 + y_{max}^2 + x_1^2 - 2x_1\sqrt{2(x_{max}^2 + y_{max}^2)} * \cos(MOA)}$

This article is written by students. It may include omissions and imperfections.

Blowing-up a Triangle

2016- 2017

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I. Our Task

Our problem concerns a random triangle, say, ABC, whose longest side is denoted by d and whose center of gravity is G. We define the inflation process as follows: starting from G, we draw a segment on which we place the point M4, obtaining the polygon E4. The distance between M4 and any vertex of our triangle cannot be longer than d. If we rotate the half-line by 1◦ and place the point M5 likewise, we obtain the polygon E5, and so on. We are required to describe the polygon E363.

II. a. Approach to the Research Topic

The first step we took towards reaching a solution to our subject was acquiring a proper understanding of our research topic. We started this process by creating manually drawn representations of the "inflated" triangle and while doing this we had the opportunity to make a series of initial observations concerning its formation.

One of these initial observations we pointed out after the first examination was that the index number 'n' used in marking both the series of points 'M' and of the polygons 'E' will play a vital role in finding the number of vertices E363 has. At first, we were inclined to believe that the number of vertices coincides with n, which turned out to be true only with regard to the first part of the process of "inflation". As our later discovery showed, the situation of the correspondence between these two will change upon reaching a certain point of the evolution of the structure.

Another observation we were able to make using our early representations was that the points M used to build the polygon E are in fact placed on three circle arcs having their center in one of the initial triangle's vertices, A, B or C, varying depending on the situation, and the radius *d*. We established the length of the radius to be *d* using the following reasoning: as it was stated in the description of the topic, a point M is placed as far as possible from the vertices of the triangle, but not further than d, the length of the longest side; based on these two ideas we understood that any line segment determined by any point M and the vertex of the triangle that is the farthest away from it (A, B or C) will always have the maximum length, in other words *d*.

For the sake of a better comprehension of the changes of the structure resulted from the variation in the shape of the initial triangle, we created a computer based representation using the mathematical program GeoGebra which allowed us to move the key points of the structure, which enabled us to conclude that the size of the circle arcs determined by the points M is indirectly proportional to the size of the vertex corresponding to its center.

Also, considering that each stage of the “inflation” process means a rotation of one degree of the initial half line starting in G on which the point M is placed upon, we realized that after 360 “inflations” we will get back at the starting M point. And so, we thought of approximating the infinite possible positions a point M could take on the circle arc corresponding to one degree, to one fixed position covering the aforementioned distance. Even though we dismissed this idea because of the lack of precision, it brought the idea of approximation in our research as a possible option, which was necessary for obtaining the final structure of E363.

II.b. Another Approach to the Research Topic

If we don’t take the center of gravity into consideration and are no longer constraint by the 1◦ rule a different form of the problem emerges. We define this new inflation process as follows: Knowing that d (the original diameter) is the greatest distance between two points that we can find in a figure. A shape is inflated when we can't add a point without increasing its diameter. So, according to this definition, an inflated triangle is a curve with a constant diameter. In the first instance, the triangle is not inflated as its diameter does not increase, while in the second picture the triangle can be inflated by adding the point D because the distance between A and D is greater than the original diameter.

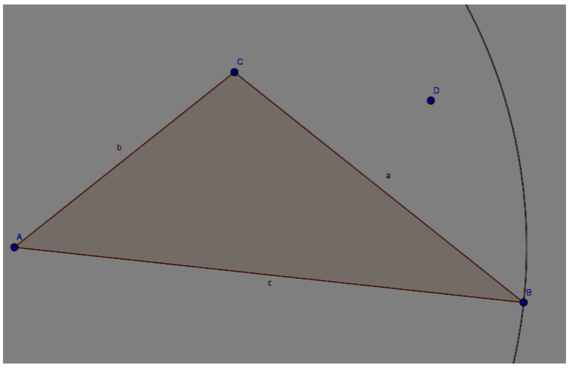

Figure 1. Non-inflated triangle

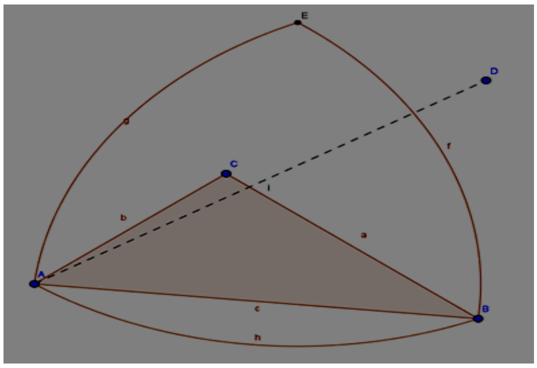

Figure 2. Inflated triangle

We have also found that all the inflated triangles resulting from the original are, when overlapped, forming a particular shape, that we called the envelope. By this we mean that, the envelope of a triangle contains all of the possibilities of inflation of that particular triangle and is actually the intersection of the 3 arcs with center in A, B respectively C with the diameter *d*. The fact that all inflated triangles are included in this envelope can be proved as every point of the envelope will increase the diameter of the initial triangle. In the following pictures, the envelope is depicted in green.

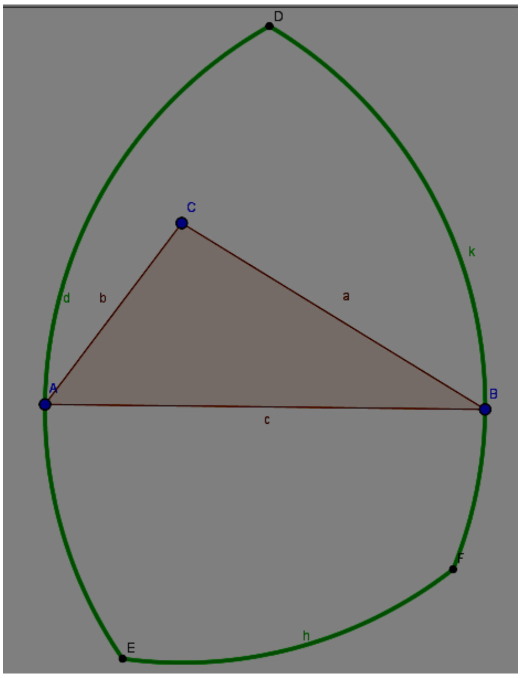

Figure 3. Envelope of the triangle

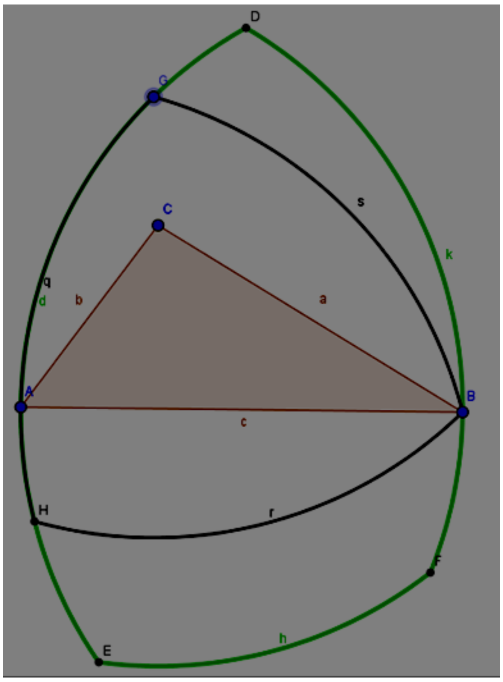

Figure 4. Envelope of the triangle

If the inflation process is limited to three arcs the triangle becomes a Reuleaux triangle as depicted in *Figure 6* while if more arcs are added it turns into a different shape as can be noticed in *Figure 5*.

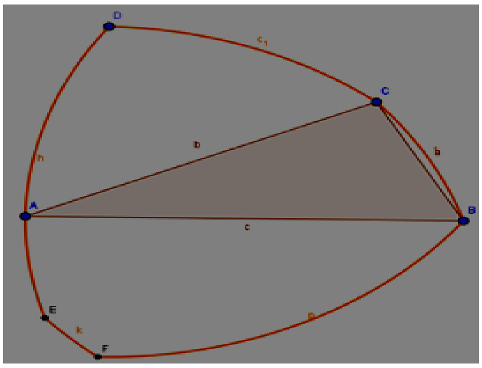

Figure 5 Inflated triangle with more than 3 arcs

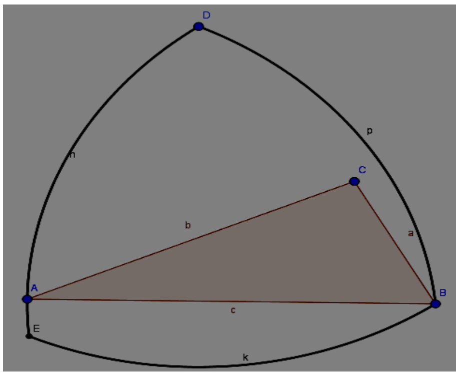

Figure 6 Inflated triangles with just 3 arcs

All the examples we tested on led us to believe that the perimeter of all inflated triangles is the same, but unfortunately, we did not find a way to prove it.

II. Description of the polygon E363

In order to reach the number of vertices and sides of E363, we had to understand how their number varies throughout the whole process of "inflation" and how it's related to the index number of M and E. As previously stated, for the first stages of the evolution of the structure, the number of vertices is equal to the index number of M, but after exceeding the intersection between a side of the initial triangle (AB, AC or BC depending on its construction) or one of the points A, B or C (considering that the points that determine the side with the length d will be placed on circle arcs themselves) the situation changes.

It is established in the research topic that the polygons E formed in the process are convex polygons. The problem appears after going past those critical points mentioned above, since if we do so in the same way we did up until that point, the newly formed polygon E will no longer be a convex one.

So, in order to maintain the convexity of the polygon we understood that we had to exclude the vertex of the initial triangle that caused this effect. In other words, even though we will place a new M after this critical point, the number of vertices will still be equal to the one of the previous stage of the "inflation". This means that the number of vertices and sides of the structure from now on will no longer be the index number, but the index number n minus one.

This situation occurs for each of the three points A, B or C, thus resulting, in the end, in the exclusion of all three of them as vertices in E363, its number of vertices becoming equal to the index number n minus 3. In this manner, we concluded that the polygon E363 has a total of 360 vertices and 360 sides.

Considering just how short the line segments between two consecutive M-s and how wide the angles determined by them are, we felt it was right to approximate the shape of E363 to the one determined by the circle arcs on which the M-s are placed. The intersection points between the circle arcs, representing one of their two ends, were in fact the points where two of the points A, B or C were equally distanced from M. We named these points switch points, given that they are situated where the trajectory that the M points follow changes.

IV. The Perimeter and the Area of E363

Another thing that we thought of when trying to describe the polygon was its perimeter and its area. In order to calculate these, we tried to build a program in C++ in which we drew a triangle and the first GM. From then on, the program would generate all the other GMs (at precisely 1^{o} from the initial segment it would build a new segment also starting from G until it intersected the outline of the triangle ABC. There it would build a point and continue from there on the same direction until the segment intersects one of the three arcs). Knowing the coordinates of the starting- and endpoints of each line it was programmed to determine the length and calculate the difference between the two consecutive M points. By adding the differences between ensuing endpoints, it would lead to the exact perimeter of the shape. Generating all the steps taken in the forming of the E363 polygon, parted the shape into 360 triangles whose three sides are all known. The program would apply the formula of Heron ($A=\sqrt{p(p-a)(p-b)(p-c)}$, where p represents the semi perimeter and a, b and c the lengths of the sides) on each small triangle formed by two successive segments and by adding the areas of all the small triangles the area of the whole inflated triangle would be obtained. Unfortunately, our program could not be run as we encountered bugs that we could not fix.

V. Particular Cases

A special case of our task would be when the triangle ABC is equilateral. In this case, the shape of the inflated triangle coincides with a Reuleaux triangle – that is a curve of constant width which can be rotated between two parallel lines without exceeding them or through rotation cover the shape of a square almost perfectly. Its perimeter is $P=\pi d$ and its area is $A=\frac{1}{2}(\pi-\sqrt{3})d^2$, where d represents in both cases the length of a side of the equilateral triangle. According to the second definition the Reuleaux triangle is the only possible inflation of an equilateral triangle.

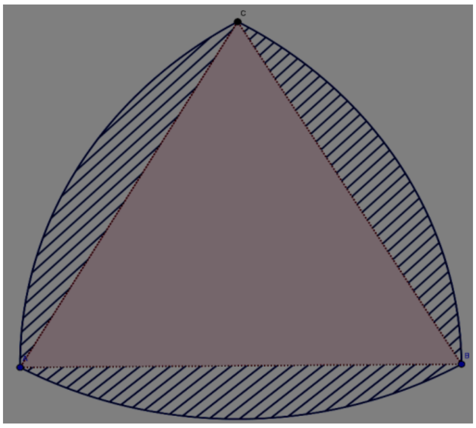

Figure 7. Reuleaux triangle

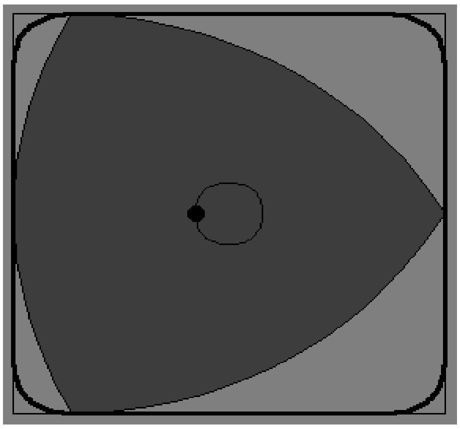

Figure 8. Reuleaux triangle

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Caution: Falling Dice!

2016- 2017

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Presentation of the research topic

Two fair dice are rolled together, repeatedly. A fair die is a die for which all sides have the same chance of appearing at every roll. We consider two events related to this random experiment:

A= the double $(6,6)$ *appears;*

B= the sum 7 *appears twice in a row.*

Here, we interpret "sum 7" as the sum of the results from both dice that appeared at one roll.

a) Determine the probability of event *A* happening in a single roll.
b) How may rolls are needed, on average, until we get the double $(6,6)$ for the first time (that is, the expected value of the number of rolls)?
c) Determine the probability of event *B* happening in exactly two rolls.
d) How many rolls are needed, on average, to get event *B* to happen for the first time?

Two friends, Anna and Bill, are playing a game based on rolling two fair dice repeatedly. Anna says that event *A* will happen before event *B* happens. Bill says the opposite of that. The dice are thrown repeatedly until one of the kids wins.

e) Who has a higher chance of winning the game? Determine Anna's odds of winning the game. For example, if her probability of winning is $\frac{5}{8}$, then her odds are $5:3$.
f) Make comments on the result based on the results of the previous points.
g) We have the information that the first roll of the dice was not the double $(6,6)$. What are now Anna's odds of winning the game?

Brief presentation of the conjectures and results obtained

This article presents an introduction to the study of probabilities, through a problem with 7 tasks. In order to approach the problem, we explain in the beginning of the article some basic properties and definitions about probability, expected values and discrete random variables. The problem requires us to find the probability that some events happen when rolling dice, or their mathematical expectancy. Our problem impresses because, although it can be seen as a children's game, it contains a lot of useful information for the field of statistics.

The text of the article
Definitions and Properties

1.1. Classical Definition of Probability. If a random experiment (process with an uncertain outcome) can result in a positive integer *n* of mutually exclusive and equally likely outcomes, and if n_A of these outcomes have an attribute *A*, then the probability of *A* is the fraction $\frac{n_A}{n}$.

1.2. Definition of incompatible events. Two events that cannot happen simultaneously are called *incompatible* or *mutually exclusive*.

1.3. Axiomatic Definition of Probability. If we do a certain experiment, which has a finite sample space Ω, we define the probability as a function that associates a certain probability, $p(A)$ with every event A, satisfying the following properties:

1.3.1. The probability of any event *A* is positive or zero. Namely $p(A) \geq 0$. The probability measures, in a certain way, the chance of event *A* to happen: the smaller the probability, the less chances for *A* to happen.

1.3.2. The probability of the sure event is 1. Namely $p(\Omega) = 1$. And so, the probability is always greater than 0 and smaller than 1: probability 0 means that there is no probability for it to happen (it is an impossible event), and probability 1 means that it will always happen (it is a sure event).

1.3.3. The probability of the union of every set of two by two incompatible events is the sum of the probabilities of the events. That is, if we have, for example events *A*, *B*, *C*, and these are two by two incompatible, then $p(A \cup B \cup C) = p(A) + p(B) + p(C)$.

1.4. Main Properties of Probability.

1.4.1. The probabilities of complementary events add up to 1:

$$p(A) + p(\bar{A}) = 1.$$

Often, we will use this property to calculate the probability of the complementary set.

This property, which turns out to be very useful, can be generalized. If we have three or more events, two by two incompatible, and such that their union is the whole space, that is to say, *A*, *B*, *C*, two by two incompatible so that $A \cup B \cup C = \Omega$, then $p(A)+p(B)+p(C)=1$. We say in this case that *A*, *B*, *C* form *a complete system of events*. Let's observe that whenever we express Ω as a set of elementary events, in fact we are giving a complete system of events.

1.4.2. The probability of the impossible event is zero:

$$p(\varnothing)=0.$$

1.4.3. If $A \subset B$, then $p(A) \leq p(B)$.

The notation "if $A \subset B$" reads "if the event *A* is included in event *B*", that is to say if all the possible results that satisfy *A* also satisfy *B*.

1.4.4. $p(A \cup B) = p(A) + p(B) - p(A \cap B)$.

1.5. **Definition of Discrete Random Variables**. A random variable is a function that assigns a numerical value to each possible outcome of a probabilistic event. A discrete random variable can take only distinct, separate values.

A probability distribution for a discrete random variable X consists of:

- All its possible values $x_1, x_2, \ldots, x_n, \ldots$
- Corresponding probabilities $p_1, p_2, \ldots, p_n, \ldots$

with the interpretation that $p(X=x_1)=p_1,\ p(X=x_2)=p_2, \ldots,\ p(X=x_n)=p_n, \ldots$. Each $p_i \geq 0$ and $p_1+p_2+\ldots+p_n+\ldots=1$.

1.6. Definition of Expected Value. The expected valuable of a discrete random variable *X* is

$$E(X)=\sum_{i=1}^{n} p(X=x_i)\cdot x_i=\sum_{i=1}^{n} p_i \cdot x_i,$$ for a finite number of possible values or

$$E(X)=\sum_{i=1}^{\infty} p(X=x_i)\cdot x_i=\sum_{i=1}^{\infty} p_i \cdot x_i,$$ for a countable infinite number of possible values.

1. Solution

a) Let *p* be the probability that event *A* happens in a single roll and let p' be the probability of getting a 6 when you roll a die once. We denote this event by *C*.

For event *A* to happen, both dice must show 6 when rolled once. So, event *C* must happen once for each of the two dice. This means that $p = p' \cdot p'$.

When rolling a die, the probability of number 6 appearing is $\frac{1}{6}$, i.e., $p' = \frac{1}{6}$.
Therefore, the probability of event A happening in a single roll is

$$p = p' \cdot p' = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}.$$

b) **Method I**

We denote by x the number of rolls necessary, on average, to get the double $(6,6)$ for the first time.
If we get $(6,6)$ on the first roll, then we would only need one roll. The probability of this happening is, according to the previous point, $\frac{1}{36}$.
In the opposite case, with a probability of $\frac{35}{36} = 1 - \frac{1}{36}$, if we do not get the double $(6,6)$ on the first roll, we would need $x+1$ rolls.
So, we have $x = \frac{1}{36} \cdot 1 + \frac{35}{36} \cdot (x+1)$, from which we get that $x = 36$.

Method II

Let X be the random variable associated to the event "A happens for the first time at roll number n".
The probability of event A happening in any one throw is, as we have established at point a), $p = \frac{1}{36}$, and the probability of event A not happening at any one throw is

$$r = p\left(\bar{A}\right) = 1 - p(A) = 1 - p = \frac{35}{36}.$$

Thus, the probability of event A happening for the first time at roll number k (where k is a positive integer) is $p \cdot r^{k-1}$. Therefore, we get that

$$X \begin{pmatrix} 1 & 2 & 3 & \dots & k & \dots \\ p & p \cdot r & p \cdot r^2 & \dots & p \cdot r^{k-1} & \dots \end{pmatrix}.$$

So, the expected value associated to the random variable X is

$$E(X) = \sum_{k=1}^{\infty} \left(k \cdot p \cdot r^{k-1}\right) = p \cdot \sum_{k=1}^{\infty} \left(k \cdot r^{k-1}\right)$$. We denote

$$S_n = \sum_{k=1}^{n} \left(k \cdot r^{k-1}\right) = 1 \cdot r^0 + 2 \cdot r^1 + 3 \cdot r^2 + \dots + n \cdot r^{n-1}.$$

That yields

$$(1-r) \cdot S_n = (1-r)\left(1 \cdot r^0 + 2 \cdot r^1 + 3 \cdot r^2 + \dots + n \cdot r^{n-1}\right) =$$

$$= 1 \cdot r^0 + 2 \cdot r^1 + 3 \cdot r^2 + \dots + n \cdot r^{n-1} - r - 2 \cdot r^2 - 3 \cdot r^3 - \dots - n \cdot r^n =$$

$$=1+r+r^2+\ldots+r^{n-1}-n\cdot r^n=\frac{1-r^n}{1-r}-n\cdot r^n=\frac{1-r^n-n\cdot r^n\cdot(1-r)}{1-r}.$$

Therefore,

$$S_n=\frac{1-r^n-n\cdot r^n\cdot(1-r)}{(1-r)^2}=\frac{1-r^n-n\cdot r^n\cdot p}{p^2}.$$

Using this fact, we can write:

$$E(X)=p\cdot\lim_{n\to\infty}S_n=p\cdot\lim_{n\to\infty}\frac{1-r^n-n\cdot r^n\cdot p}{p^2}=\frac{1}{p}\cdot\lim_{n\to\infty}\left(1-r^n-n\cdot r^n\cdot p\right).$$

Because $r\in(0,1)$, we get $E(X)=\frac{1}{p}\cdot\lim_{n\to\infty}\left(1-r^n-n\cdot r^n\cdot p\right)=\frac{1}{p}\cdot(1-0-0)=\frac{1}{p}=36$.

c) Let q be the probability of event B happening in exactly two rolls and s be the probability than we get the sum 7 when rolling two dice. We denote this event by S (success).
For event B to happen, event S must happen twice in a row. That implies that $q=s\cdot s$.
When rolling two dice, we get one of 36 results from the set $\left\{(x,y)\,\middle|\,x,y=\overline{1,6}\right\}$.
The only favourable results (which have sum 7) are

$$\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}.$$

Therefore, $s=\frac{6}{36}=\frac{1}{6}$, from where we get that the probability of event B happening in exactly two rolls is $q=s\cdot s=\frac{1}{6}\cdot\frac{1}{6}=\frac{1}{36}$.

d) **Method I**

We denote by y the number of rolls that are necessary, on average, until event B happens for the first time.
If we do not get sum 7 on the first roll, event that happens with the probability of $1-s=1-\frac{1}{6}=\frac{5}{6}$, then we would need $y+1$ rolls.
Else, if we get sum 7 on the first roll, but not on the second roll, then $y+2$ rolls would be needed. The probability of this event is $s\cdot(1-s)=\frac{1}{6}\cdot\frac{5}{6}=\frac{5}{36}$.
Lastly, if we get sum 7 on the first two throws, then we would need 2 rolls.
Thus,

$$y=\frac{5}{6}\cdot(y+1)+\frac{5}{36}\cdot(y+2)+\frac{1}{36}\cdot 2,$$

which implies $y=42$.

Method II

Let Y be the random variable associated to the event "B happens for the first time on roll number n". The possible values of Y are $\{2, 3, 4, \ldots\}$. If we denote by p_k the probability of B happening for the first time on roll k, then we have

$$Y\begin{pmatrix} 2 & 3 & \ldots & k & \ldots \\ p_2 & p_3 & \ldots & p_k & \ldots \end{pmatrix},$$

where $\sum_{k=2}^{\infty} p_k = 1$.

$s = \frac{1}{6}$ is the probability of getting sum 7 when rolling two dice, event which we denoted by S (success), and let $f = 1 - s = \frac{5}{6}$ be the probability of not getting sum 7 when rolling two dice, event which we denote by F (failure).

We observe that $p_1 = 0$, $p_2 = s^2$ and $p_3 = f \cdot s^2$. By considering all the possible outcomes of the event B in the first two rolls, we shall get a recurrence relation for p_k, where $k \geq 3$. Apart from the case $k = 2$, when we start with SS, for cases $k \geq 3$ we start either with F or with SF. That yields $p_k = f \cdot p_{k-1} + s \cdot f \cdot p_{k-2}$, for all $k \geq 3$.

The expected value of the number of rolls needed until event B happens for the first time is $E(Y) = \sum_{k=2}^{\infty} k \cdot p_k$.

For all $k \geq 3$ we have $k \cdot p_k = f \cdot (k-1+1) p_{k-1} + s \cdot f \cdot (k-2+2) p_{k-2}$. Therefore, we have that

$$\sum_{k=3}^{\infty} k \cdot p_k = f \cdot \left(\sum_{k=3}^{\infty} (k-1) \cdot p_{k-1} + \sum_{k=3}^{\infty} p_{k-1} \right) + f \cdot s \cdot \left(\sum_{k=3}^{\infty} (k-2) \cdot p_{k-2} + 2 \sum_{k=3}^{\infty} p_{k-2} \right)$$

$$\Rightarrow E(Y) - 2 \cdot p_2 = f \cdot \left(\sum_{k=2}^{\infty} k \cdot p_k + \sum_{k=2}^{\infty} p_k \right) + f \cdot s \cdot \left(\sum_{k=1}^{\infty} k \cdot p_k + 2 \sum_{k=1}^{\infty} p_k \right)$$

$$\Rightarrow E(Y) - 2 \cdot s^2 = f \cdot (E(Y) + 1) + f \cdot s \cdot (E(Y) + 2)$$

$$\Rightarrow E(Y) \cdot (1 - f - f \cdot s) = 2 \cdot s^2 + f + 2 \cdot f \cdot s$$

$$\Rightarrow E(Y) \cdot (1 - (1-s) - (1-s) \cdot s) = 2 \cdot s^2 + (1-s) + 2 \cdot (1-s) \cdot s \Rightarrow E(Y) \cdot s^2 = s + 1 .$$

So, $E(Y) = \frac{1}{s} + \frac{1}{s^2} = 6 + 36 = 42$.

e) Let p_A be the probability that Anna wins the game, i.e., event *A* happens before event *B* when throwing two dice repeatedly. We observe that events *A* and *B* cannot happen simultaneously, and neither can *A* and *S* (sum 7).

If, on the first roll, event *A* happens, then the probability that *A* happened before *B* is 1 (it is a sure event). This happens with a probability of $p = \frac{1}{36}$.

If we have sum 7 on the first roll and then event *A* happens on the second turn, then the probability of *A* happening before *B* is 1, because *A* happened and *B* did not. This case has the probability $s \cdot p = \frac{1}{6} \cdot \frac{1}{36}$ of happening.

Else, if we get sum 7 on both the first two rolls, i.e., *B* happens in the first two rolls, event whose probability is $s \cdot s = q = \frac{1}{36}$ then the probability of *A* happening before *B* would be 0.

In the case (with probability $s \cdot (1 - p - s) = \frac{1}{6} \cdot \frac{29}{36}$) that we get sum 7 on the first turn, and then neither sum 7 nor *A* happen on the second turn, the probability of *A* happening before *B* is p_A, since the first two turns did not affect the game (we didn't make any steps toward *B* or *A* happening).

Also, in the case (with probability $1 - p - s = \frac{29}{36}$) that neither sum 7 nor *A* happen on the first turn, the game was not affected by the first turn. That means that the probability that *A* happens before *B* in the rolls that follow is p_A.

All of these cases, combined, form a complete system of events.

Therefore, we have that

$$p_A = \frac{1}{36} \cdot 1 + \frac{1}{6} \cdot \frac{1}{36} \cdot 1 + \frac{1}{36} \cdot 0 + \frac{1}{6} \cdot \frac{29}{36} \cdot p_A + \frac{29}{36} \cdot p_A.$$

Thus, $p_A = \frac{7}{13}$, which means that Anna's odds of winning are $7:6$.

f) Although both events *A* and *B* have the same chance of occurring (that is, $\frac{1}{36}$), when conditioning them, one conditioned by the other one, the event *A* has more chances to occur before event *B* has occurred.

However, this result is expected, as event *B* has a chance of occurring if at least two rolls of dice are made, while event *A* can occur even in one roll.

g) **Method I**

Knowing that the first roll of the dice was not the double $(6,6)$, the results we can get are from the set $\{(x, y) \mid x, y = \overline{1,6}\} \setminus \{(6,6)\}$, which has 35 elements. The results from this set that have sum 7 are

$$\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}.$$

We get that the probability that on the first roll we get sum 7 is $s' = \frac{6}{35}$. The probability that we do not is $f' = 1 - s' = \frac{29}{35}$.

If we did not get sum 7 on the first turn, the game was not affected by the first turn. That means that the probability that *A* happens before *B* is still $p_A = \frac{7}{13}$, as we have proved at point e).

Else, if we get sum 7 on the first turn, we have three cases: either we get sum 7 on the second turn, or we get $(6,6)$, or we do not get any of these. In the first case (with probability $s = \frac{1}{6}$), *B* would happen in the second turn, and *A* would not have happened by then. So, the probability that event *A* happens before event *B* does would be 0. In the second case (with probability $s = \frac{1}{36}$, *A* would happen before *B*, thus Anna wins the game with a probability of 1 (a sure event). In the third case (with probability $1 - p - s = \frac{29}{36}$), the game would not be affected by the first two turns, because we did not make any steps toward *A* or *B* happening. Thus, the probability of *A* happening before *B* is $p_A = \frac{7}{13}$, from point e).

All of these cases, combined, form a complete system of events.

Therefore, we have that the probability that event *A* happens before event *B*, knowing that the first roll of the dice was not the double $(6,6)$ is

$$p_A{}' = \frac{29}{35}\cdot\frac{7}{13} + \frac{6}{35}\cdot\frac{1}{6}\cdot 0 + \frac{6}{35}\cdot\frac{1}{36}\cdot 1 + \frac{6}{35}\cdot\frac{29}{36}\cdot\frac{7}{13} = \frac{239}{455}.$$

Thus, Anna's odds of winning the game if we know the first roll of the dice was not the double $(6,6)$ are $239:216$.

Method II

Let *M* be the event that Anna wins the game, and let *N* be the event that the first roll is not the double $(6,6)$.

Then,

$$M = M \cap \left(N \cup \bar{N}\right) = (M \cap N) \cup \left(M \cap \bar{N}\right)$$

and so, as $(M \cap N) \cap \left(M \cap \bar{N}\right) = \varnothing$, we can write:

$$p(M) = p(M \cap N) + p\left(M \cap \bar{N}\right).$$

Denote by $M \big| N$ the event that Anna wins the game knowing that the first roll was not $(6,6)$.

Then, $M \big| \bar{N}$ the event that Anna wins the game knowing that the first roll was $(6,6)$.

By the definition of a conditional probability, $p\left(M\middle|N\right)=\dfrac{p(M\cap N)}{p(N)}:=p'_A$, the required probability here. Thus,

$$p(M)=p\left(M\middle|N\right)\cdot p(N)+p\left(M\middle|\bar{N}\right)\cdot p\left(\bar{N}\right)$$

$$\Rightarrow p_A=p'_A\cdot(1-p)+1\cdot p$$

$$\Rightarrow \frac{7}{13}=p'_A\cdot\frac{35}{36}+1\cdot\frac{1}{36}\Rightarrow p'_A=\frac{239}{455}.$$

Conclusions

We have answered all the points of the proposed problem. Some of the tasks where solved using more than one method. By solving this problem, we have enlarged our knowledge on probability and random variables. Some of the requirements were very challenging and they required advanced skills in probability. The problem gave us a short insight into the very popular world of betting.

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Cryptology

2016- 2017

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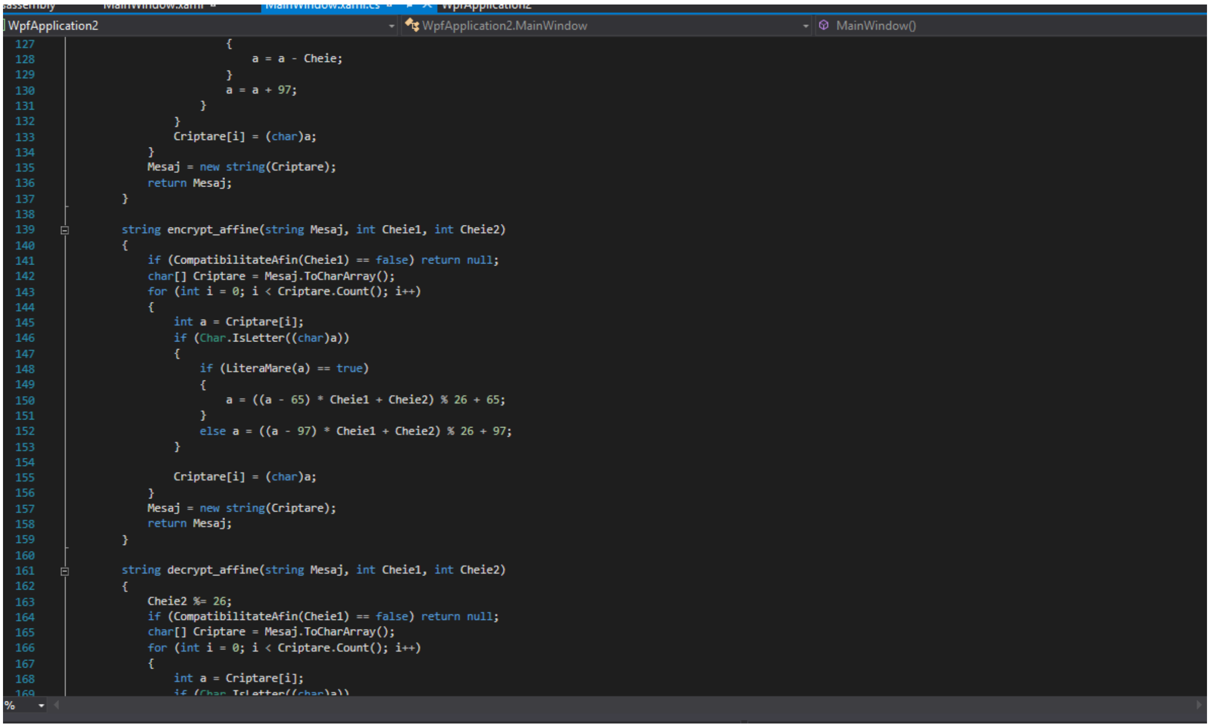
We studied for our project different methods of encrypting messages, especially Caesar’s Code and the Affine Code.

After studying our encrypting methods and learning how they work, our team started including them in computer programs resulting into two programs, one has the Caesar’s Code and the other has the Affine Code. For now, we managed on merging the programs and we succeeded making them work together. In the near future, we think about adding more encrypting methods to our program.

```
WpfApplication2                    WpfApplication2.MainWindow                    MainWindow()
        string encrypt_cezar(string Mesaj, int Cheie)
        {
            char[] Criptare = Mesaj.ToCharArray();
            for (int i = 0; i < Criptare.Count(); i++)
            {
                int a = (int)Criptare[i];
                if (Char.IsLetter((char)a))
                {
                    if (LiteraMare(a) == true)
                    {
                        a = (a - 65 + Cheie) % 26 + 65;
                    }
                    else a = (a - 97 + Cheie) % 26 + 97;

                }
                Criptare[i] = (char)a;
            }
            Mesaj = new string(Criptare);
            return Mesaj;
        }

        string decrypt_cezar(string Mesaj, int Cheie)
        {
            Cheie %= 26;
            char[] Criptare = Mesaj.ToCharArray();
            for (int i = 0; i < Criptare.Count(); i++)
            {
                int a = Criptare[i];
                if (Char.IsLetter((char)a))
                {
                    if (LiteraMare(a) == true)
                    {
                        a = a - 65;
                        if (a - Cheie < 0)
                        {
                            a = 26 - Cheie + a;
                        }
                        else
                        {
                            a = a - Cheie;
                        }
                        a = a + 65;
```



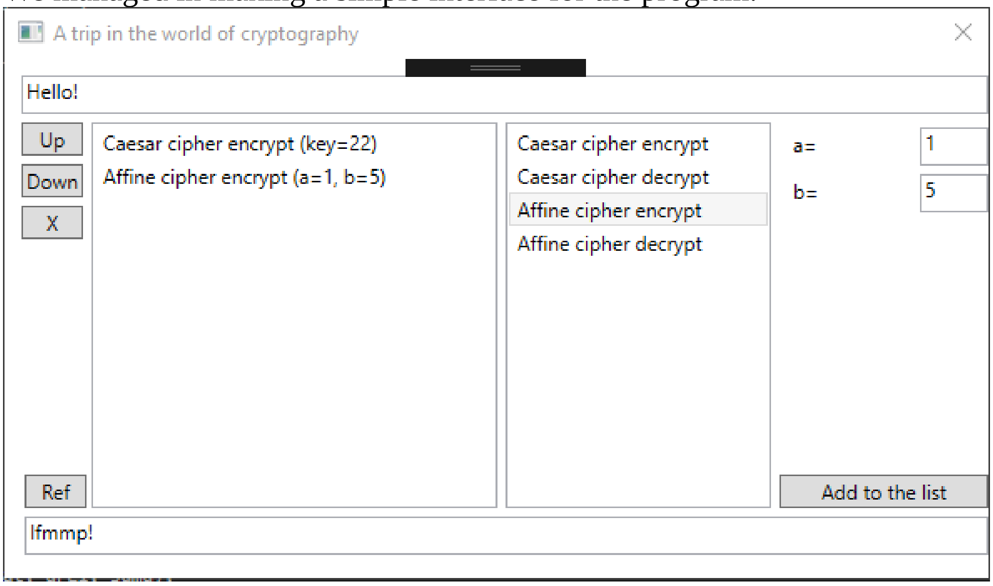
```
                    {
                        a = a - Cheie;
                    }
                    a = a + 97;
                }
            }
            Criptare[i] = (char)a;
        }
        Mesaj = new string(Criptare);
        return Mesaj;
    }

    string encrypt_affine(string Mesaj, int Cheie1, int Cheie2)
    {
        if (CompatibilitateAfin(Cheie1) == false) return null;
        char[] Criptare = Mesaj.ToCharArray();
        for (int i = 0; i < Criptare.Count(); i++)
        {
            int a = Criptare[i];
            if (Char.IsLetter((char)a))
            {
                if (LiteraMare(a) == true)
                {
                    a = ((a - 65) * Cheie1 + Cheie2) % 26 + 65;
                }
                else a = ((a - 97) * Cheie1 + Cheie2) % 26 + 97;
            }

            Criptare[i] = (char)a;
        }
        Mesaj = new string(Criptare);
        return Mesaj;
    }

    string decrypt_affine(string Mesaj, int Cheie1, int Cheie2)
    {
        Cheie2 %= 26;
        if (CompatibilitateAfin(Cheie1) == false) return null;
        char[] Criptare = Mesaj.ToCharArray();
        for (int i = 0; i < Criptare.Count(); i++)
        {
            int a = Criptare[i];
```

We managed in making a simple interface for the program.



A. <u>Caesar's code:</u>

1. The Latin historian Suetonius Tranquillus Gaius (c. 69-122), in his ”De Vita Caesarum”, included hints about the secret mode of communication used by Emperor Julius Caesar and his generals to protect messages of military significance.
2. Caesar’s cipher or Caesar’s code is a simple method of encrypting messages based on the substitution of the letter, from the original text, occupying the position n in the alphabet with the letter that occupies the position n+3.
3. The message codification scheme based on Caesar’s Code in the Latin alphabet with 26 letters.

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W

4. Using the Latin alphabet with 26 signs and Caesar's Cipher, the keys for encryption/decryption are:

   Encryption function $E_n(x) = (x+n) \bmod 26$

   Decryption function $D_n(x) = (x-n) \bmod 26$

5. Example of encryption using the Latin alphabet with 26 letters and the original Caesar Code ($x+3$)

VIVE LA FRANCE	→	S	F	S	B	I	X	C	O	X	K	Z	B
MATH EN JEANS	→	J	X	Q	E	B	K	G	B	X	K	P	

B. <u>Affine encryption system:</u>

1. Affine encryption system is a generalization of the monoalphabetic system.
2. We consider the Latin alphabet with 26 symbols and $K = \{(a,b) \mid a,b \in Z_{26}, cmmdc(a,26) = 1\}$.
3. For the key $K = (a,b)$, encryption/decryption functions are:

   Encryption function $E_k(x) = ax + b \pmod{26}$

   Decryption function $D_k(y) = a^{-1}y + a^{-1}(26-b) \pmod{26}$

4. The application of encryption function for all 26 letters of the alphabet corresponds to the table below, where the first two rows are related to the text and the last two lines are related to the text encrypted:

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
3	10	17	24	5	12	19	0	7	14	21	2	9	16	23	4	11	18	25	6	13	20	1	8	15	22
D	K	R	Y	F	M	T	A	H	O	V	C	J	Q	X	E	L	S	Z	G	N	U	B	I	P	W

Notes: mod – is the rest of the division
cmmdc – is the gcd (greatest common divisor)

Until we meet at the conference held in Cluj, Romania, our team keeps working and improving the computer programs and the presentation so that we will bring you an even better and more interesting experience.

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http://www.galaxyng.com/adrian_atanasiu/cursuri/cript/c2.pdf

This article is written by students. It may include omissions and imperfections.

Distance Between Words

2016- 2017

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The research topic

We have a dictionary in which the words consist of five letters belonging to the set consisting of the first 8 letters of the alphabet {A, B, C, D, E, F, G, H}. We have to define a distance, denoted d, between two words in the dictionary. A distance is a function that assign to each pair of words a real positive number and satisfies the following axioms:

- Symmetry: for each X and Y in the dictionary, d (X, Y) = d (Y, X);
- For each X and Y in the dictionary, d (X, Y) = 0 => X=Y;
- The triangle inequality:

d (X, Z) <= d (X, Y) + d (Y, Z)
for all X, Y, Z words in the dictionary.

Results

At the end of our research we offered:

- Three different descriptions of the distance
- The verification of the axioms
- A program written in *C++* that calculate the distance between two words

Our research

First Approach

The first attempt to define a distance on a set of words, that is a dictionary, is the following: We associate to each word a natural number, called the rank of the word, indicating its position in the dictionary ordered lexicographically.
The distance between two words can be defined as the absolute value of the difference of their ranks. A practical method to determine the distance between two different words is to count the words lying between them, and to take the successor of the obtained number.
From the definition above it follows immediately

d(X,Y)=0 if and only if X=Y.
If n is the rank of the word X and m is the rank of the word Y, then d (X, Y) = |n-m| = |m-n| = d (Y, X).
If n, m and p are the ranks of the words X, Y, respectively Z, then we have the inequality
d (X, Y) = |n-m| <= |n-p| + |p-m| = d (X, Z) + d (Z, Y).
For a word which has n letters, we define a method of giving the value of the word's rank, in which every letter has its own order.
For example, in the word $\overline{a_n a_{n-1} .... a_1}$,a_1 has the order 1, a_2 has the order 2, a_n has the order n. The rank of $a_n a_{n-1} \ldots a_2 a_1$ will be the sum of the number given by the function d2(x) to each letter.
d1(a)= {
1, x='A'; 2, x='B'; 3 , x='C' ; 4 , x='D' ; 5, x='E'; 6 , x='F'; 7 , x='G' ; 8 , x='H' ; 9 , x='I' ; 10 , x='J' ;
11, x='K'; 12, x='L'; 13, x='M'; 14, x='N'; 15, x='O'; 16, x='P'; 17, x='Q'; 18, x='R'; 19, x='S'; 20, x='T';
21, x='U'; 22, x='V'; 23, x='W'; 24, x='X'; 25, x='Y'; 26, x='Z';
}.
D2(a_n)=d1(x)*26^(n-1)
As a result, the rank of the word $\overline{a_n a_{n-1} .... a_1}$ will be equal to $\sum_{i=1}^{i \to n} d2(a_i)$.
The distance between two different words will be the absolute value of the difference between their ranks.

Second Approach

The idea is to mimic the octal representation of natural numbers. Recall that in the decimal representation, a value of a number is computed as a sum between the unit digit multiplied by $1= 10^0$ plus the tens digit multiplied by 10^1 and so on.
Ex: $[100]_{10}=0* 10^0 + 0 * 10^1 + 1 * 10^2=100$.
In a similar manner in octal representation, for obtaining the value of a number we multiply the units, tens, hundreds etc. by successive powers of 8.
Example: $[100]_8=0*8^0 + 0*8^1 + 1*8^2=[64]_{10}$, that is the octal number 100 is the same as the decimal number 64.
Note that digits 8 and 9 do not exist in base 8.
In order to define a distance, we assign to every letter an octal digit: A=0, B=1, C=2 etc. (recall that we have 8 letters).
Next, we understand a word as an octal representation, and we compute the natural number associated to it in the base 8 as before.
Example: AAAAA is $0*8^0+0*8^1+\ldots+0*8^4=0$;
CCCCC is $2*8^0+ 2*8^1+2*8^2+2*\hat{}8^3+2*8^4=6409$.
The distance between two words is the absolute value of the difference between the numbers associated with that words.

Example: d (AAAAA, CCCCC) = |9362 – 0|= 9362.
Note: The last word in the dictionary is HHHHH for which the corresponding natural number is 32 767. This means that our dictionary includes 32 768 words.

Third Approach

The last way we chosen to solve our problem is the following:
For two words X and Y we count the number of positions for which the letter in X on that position is different of the letter in Y.
Ex. d(ABCDE, ABCDF) = 1; d(ABCDE, AFCGH) = 3 etc.
In this way, we take into consideration not only the letters of a word but also the position on which these letters occur.
Comparing this third approach with the previous two, we can see that it really depends on the words. In the previous cases, the distance between two words depends only on the natural numbers assigned to them (in a way or in another).

1. The defined distance is always positive. Moreover, d(X, Y)=0 if and only if X=Y, because another way to say that X=Y is to say for any position, the letter in X on those position is the same as the letter in Y.
2. The distance is symmetric: d(X,Y) = d(Y,X) because the number of positions in X which are different of the corresponding position in Y is the same as the number of positions of Y which are different of the corresponding position of X.
3. The distance satisfies the triangle inequality:
   d (X, Z) <= d (X, Y) + d (Y, Z).

The biggest distance

Since our words have 5 positions, the biggest distance between two words is 5, and it is obtained exactly when the two words have no letter on the same position.
ex. d(ABCDE, FDAGH) = 5.
For a more general case, in a dictionary containing words with the same length, the biggest distance is equal to this length.

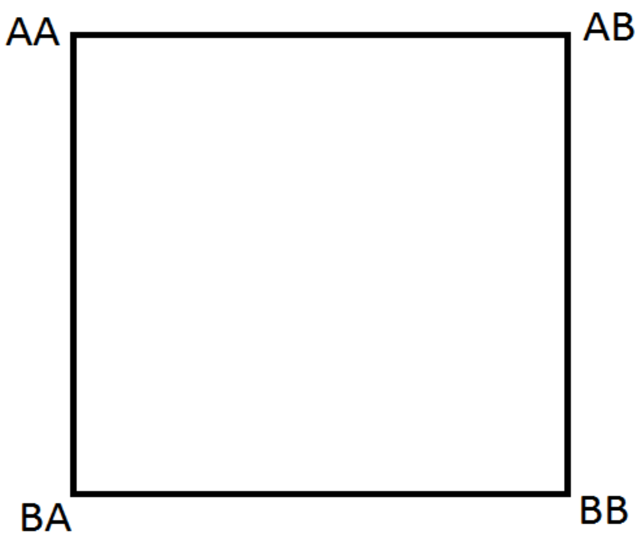

Image 1. Representation of the distance between words as a square

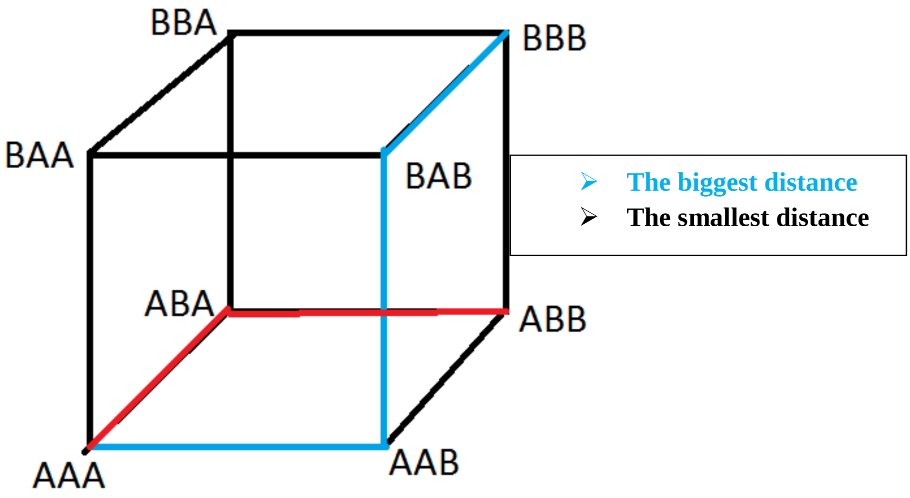

Image 2. Representation of the distance between words as a cube

Distance between mirror-words

The mirror-word is the word in which the letters are inverted on symmetrical positions. ex. The mirror-word of ABCDE is EDCBA and

d(ABCDE, EDCBA) = 4.

In our dictionary, the distance between a word and its mirror-word is 4 if and only if it is a word with distinguished letters. Therefore, we can characterize words with distinguished letters in terms of the distance and of the mirror words

Remark that the maximum of the distance between a word and its mirror-word depends on the number of positions n. More precisely:

If n is even, then d_{max}=n;

Ex. n = 6, d_{max}=d(ABCDEF, FEDCBA) = 6.

If n is odd d_{max}= n-1, because the center letter does not change position.

Ex. n = 7, d_{max}=d(ABCDEFG, GFEDCBA)=6 = n-1.

Distance 4

If X and Y are two words made of three letters d4(X, Y) = the number of the movements which takes to build Y from X . A movement is a switch of two letters or the replacement of it.

Example:

d4(BAC, BAE) = 1 We replace the C by E

d4(BAC,BCA) = 1 Switch A and C

d4(BAC, CBA)=2 Switch B and C: CBA becomes BCA then C et A : BCA becomes BAC

The program

The program was built on the first approach model, so it uses words which can have any letter from the alphabet and maximum 12 letters. The alphabet is a generalized one, and it contains 26 letters. The “ss” function calculates the ranks of the words and “main” function calculates the distance between the words, by making the difference of their ranks.

```
int ss (char e[30])
{
   char d[27];
   d[1]='A';d[2]='B';d[3]='C';d[4]='D';d[5]='E';d[6]='F';d[7]='G';d[8]='H';d[9]='I';d[10]='J';d[
11]='K';d[12]='L';d[13]='M';d[14]='N';d[15]='O';d[16]='P';d[17]='Q';d[18]='R';d[19]='S';d[20
]='T';d[21]='U';d[22]='V';d[23]='W';d[24]='X';d[25]='Y';d[26]='Z';
long long s=0;
int iiii=0;

while(e[iiii])
{
   iiii=iiii+1;
}
  for(int i=1;i<=iiii;i++)for(int j=1;j<=26;j++)
{
      if(e[i]==d[j])s=s+(pp(iiii-i)*j);
 }
  return s;
}

int main()
{
   int s,i,j,k,z,l,cc;

char b[10],e[10];
cin>>b;
cin>>e;
cout<<ss(e)<<endl<<ss(b);
cout<<modul(ss(b),ss(e))<<endl;
          }
```

This article is written by students. It may include omissions and imperfections.

Eco-math

2016- 2017

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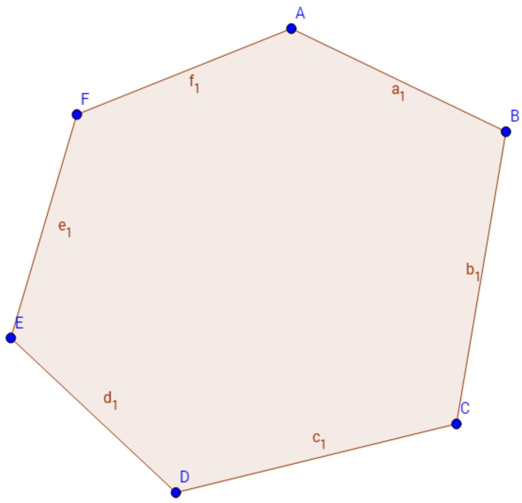
The research topic

"We are given a rectangular park, of length "x" and "y", and a number of "k" trees , with the diameter of the roots "r", growing in this park. Can we find a place where we can sink a well, with the radius of "f", without damaging the roots of the trees?"

Now, this is the problem with which we started…and we thought, "Pffff, it's too easy". So, we would like to show you our vision:

"We are given "n" points in plane (each with their own set of coordinates x and y) that will describe a convex polygon. What is the maximum number of trees (a tree has the diameter of "r") we can plant in this park, so we can be able to place a fountain (with the radius of "f") somewhere, without damaging the roots."

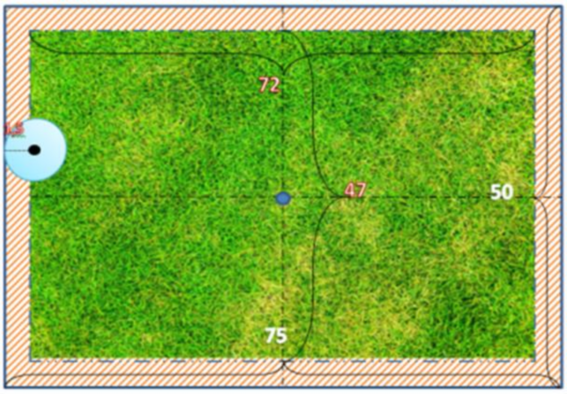


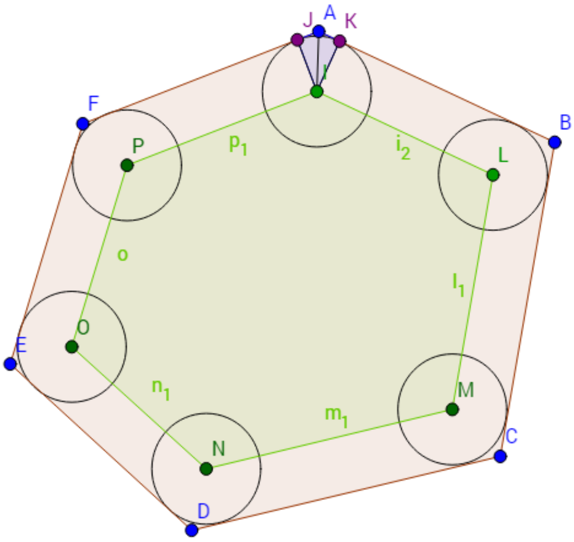
Now this is a bit more complicated but we'll walk you through it. We will present you examples from the base problem **at the** beginning so you can get used to our method. Even more we thought of a real situation:

In a park with the shape of a rectangle (50 by 75m^2) grow 200 trees with the diameter of their roots equal to 1,4 meters. In order to improve the look of the park, it is wished that a fountain, with the diameter of 3 meters, should be built somewhere. Find out if the fountain can be built between the trees without damaging their roots.

The plan:
At first, you would say that we need to calculate the area of the polygon and divide it by the area of a tree.
Well, you would be right with the first statement, but until then, we need to talk about "design". Now, imagine this: the park is surrounded by a wooden fence. You chose to put the fountain right next to the fence…but, you need to destroy it to build the fountain, or you build the fountain underneath the fence. Both of these options will ruin the look of the park. So, what do you do?

We called it “edging in”.

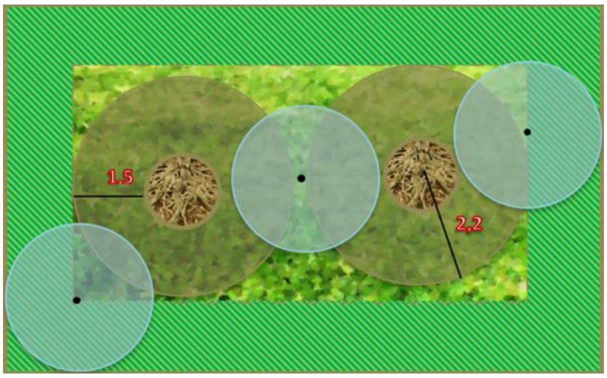


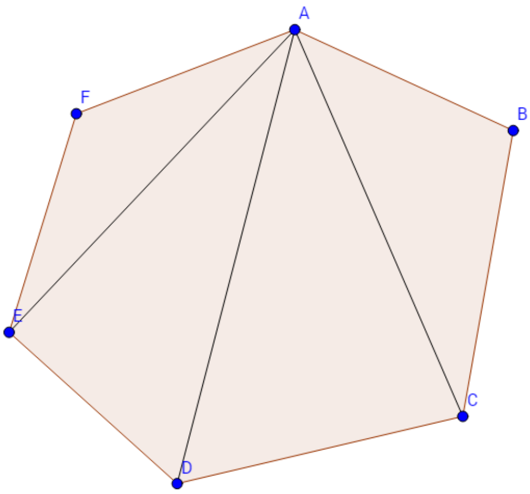


Because we can’t have the fountain merging with the fence, we need to take an imaginary line and place it parallel with the margin of the polygon, with the distance between them equal to the radius of the fountain. We repeat the same move for all the margins, and we get a smaller area.

Now, anywhere we place the center of the fountain within that smaller area, it doesn’t touch the fence. One problem solved!

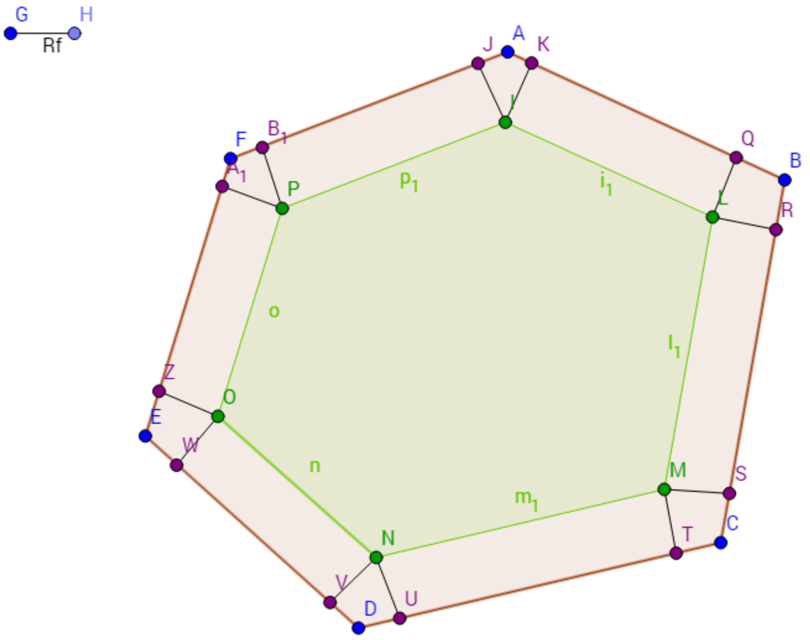
But another one rises: what if there are 2 trees near each other? We apply the same principle but now we are “edging out”, because we don’t want to cut them down. (We make an imaginary ring around each tree, adding to the diameter of the tree the fountain diameter).



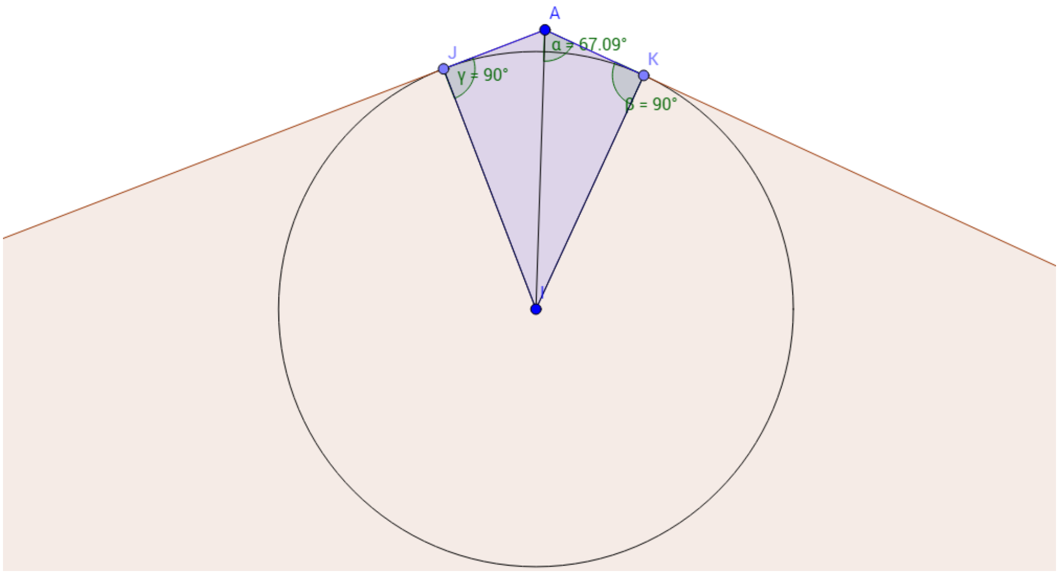


Don’t get excited because there’s a little more to it. We did the edging, now we need to find out the area of the smaller terrain. We can’t calculate it in one go, so we thought: “Why don’t we subtract the edged in area form the area of the park?” So that’s what we did.
The area of the park is easy to calculate, we divide the polygon in triangles and we calculate their surface with Heron's equation, and we add them together.

We saw that the edged in area was made of rectangles and weird shapes that we named “bird beaks”.



If you look closely, we can see that the bird beaks are actually 2 right triangles put one next to the other.



We can calculate all of the element of those triangles by finding the cosine of the angle that forms the base of the “beak” (we break the polygon into consecutive triangles, for example, if we take the polygon ABCDEF, we break it into ABC,BCD,CDE, and so on, and in each of these triangles we apply the cosine equation) , we calculate the sine, cosine and the tangent of half of that angle, and then we can calculate everything we want. From here we are a couple of steps away from finishing: we calculate the edged in area, we subtract it from the larger area, and find out the edged in surface.
All we have to do now is to divide the edged in area by the edged-out area of the tree, take the integer value of the result, and here we are, at the end of the long road.

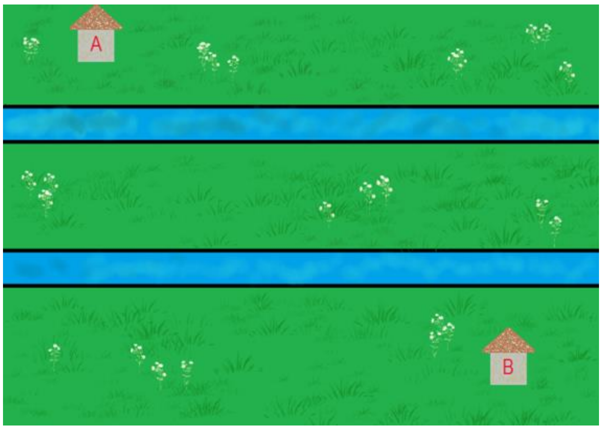
The original problem was the one we mentioned in the beginning, which was easy. We applied the same method of edging, but the edged in area we could've calculated it with ease, because we needed to subtract a diameter of the fountain from each of the margins of the rectangle. In the end, we needed to compare the edged in area with the sum of areas of all "k" edged out trees. If the sum was smaller than (or equal to) the edged in surface, we could sink a fountain without damaging the trees.

We came up with another interesting situation:

We are ecologists and we would like to present you a problem that our team has recently encountered. Not too long ago we had to build a road between two cities, A and B.
Of course, we know that people tend to travel a lot by car and pollution represents a general problem in the whole world, so we want the road to be as short as possible to decrease the harmful effects of pollution.
In building our road we met two problems, two rivers to be more precise. We decided to stick to our plan of making the road as short as possible, but because the materials we are using are quite expensive, the bridges will have to be placed perpendicular to the river so we wanted to find the best solution for the environment, which we will show you in just a few moments.

Request

Two cities are separated by two parallel rivers with the exact same width and the same distance between the shores.
Find the shortest road considering that the bridges can only be placed perpendicular to the rivers.



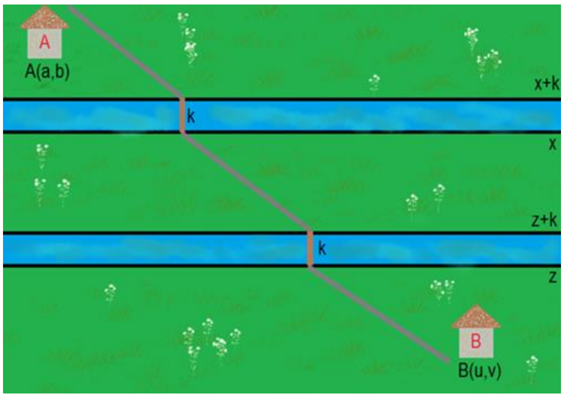
Ways of solving the problem

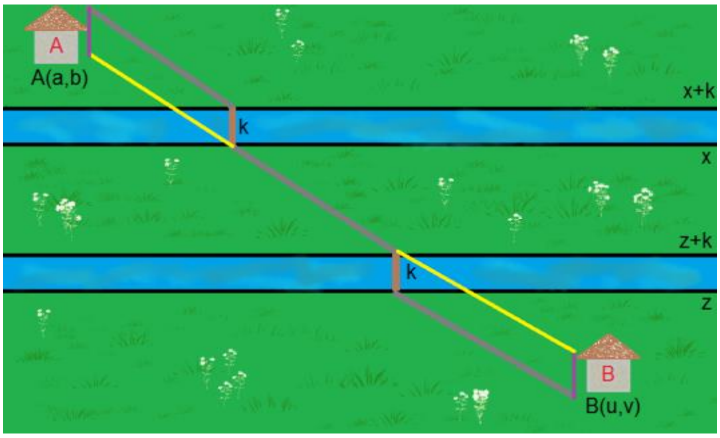
When solving this problem, we have two ways of doing it:

- The first one is to calculate each distance, but we will probably end up with some really big numbers and we will lose ourselves in the calculus.
- We, on the other way, have thought of a much easier and way more efficient method of solving the problem.

The steps of solving the problem

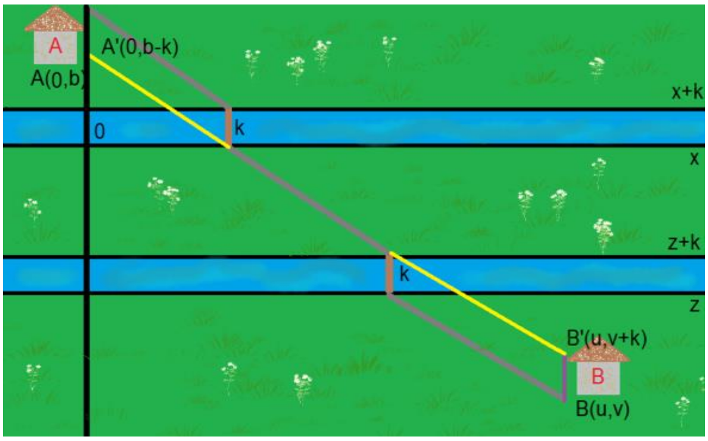
In the beginning, we are going to place the bridges in two almost random places and we are going to write down the coordinates of the cities and the rivers.

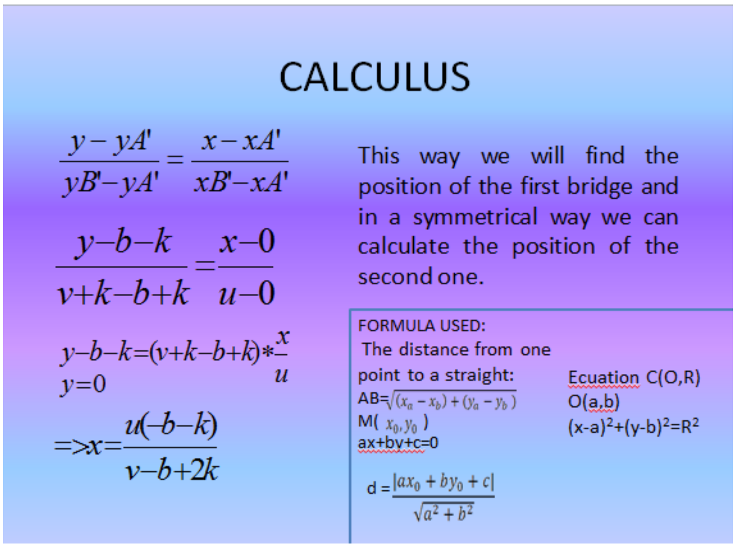




Then we are going to place the symmetrical points of the two cities that are at a distance equal to the length of the bridges from which we are going to draw parallel lines so we can form only one diagonal line, representing the shortest distance we can have.

<u>The last step in achieving our final goal</u> is to consider that in a coordinates system(x0y) A' and B' are placed at the coordinates (0, b-k) respectively (u, v+k), where k represents the length of the bridges. The first bridge will be placed at the intersection of our line and the lower shore of the first river and at the intersection with the upper shore of the second river.





<u>Special cases</u>

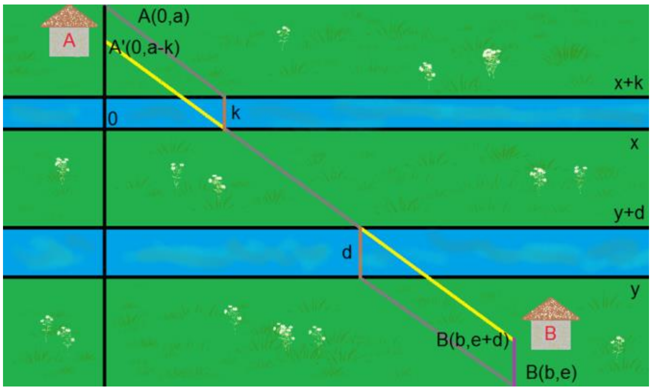
Next on our list is to study some special cases of our problem.

<u>The first case:</u> What if one of the rivers is wider than the other?

<u>The second case:</u> What if instead of a river we have a lake?

<u>The first case</u>

In case the second river is wider than the first one, the principle of solving the problem is the same and so is the calculus, we just have to change some coordinates.



<u>The second case</u>

We will consider the first city A to be located at the coordinates (0,8) and the second city b at the coordinates (8.0). The first river is placed between the lines of equation y=6 and y=7 and the lake has it's centre in the point O(4,3) and it's radius is equal to 1.

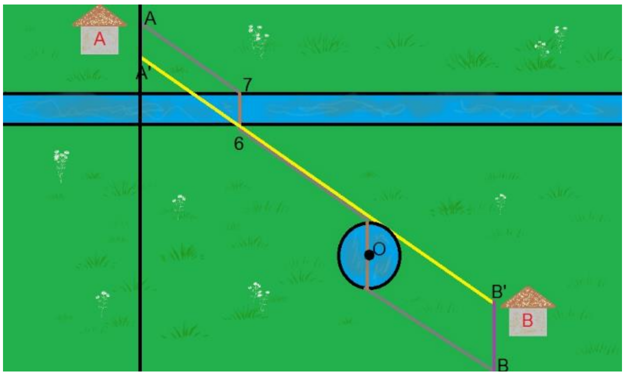
The equation of the circle (centre O and diameter equal to 2) is : $(x-4)^2 + (y-3)^2 = 1$

The symmetrical of A is A'(0;7) and the symmetrical of B is B'(8,2).

The equation of A'B' is $\frac{x-0}{8-0} = \frac{y-7}{2-7}$ => 8y-5x-56=0; We can calculate the distance from the center of the circle to the A'B' line using the formula $d = \frac{|ax+by+c|}{\sqrt{a^2+b^2}}$. In a general case M (x, y) is the point of reference, the centre of the lake, and d is the distance from M to the line of equation ax + by + c=0, but in our case the centre of the lake is represented by the point O(4,3) and the line is A'B'.

$$d = \frac{|-5x+8y-56|}{\sqrt{5^2+(-8)^2}} = \frac{|-24+20-56|}{\sqrt{25+64}} \sim 5{,}5$$

As we decided, the radius of the lake is 1. If we compare that to the result, we see that 1<5,5, which tells us that it is not necessary to build a bridge over the lake. If the distance d was shorter than the radius of the lake, the shortest road would need to cross the body of water, and a bridge would have to be built.



Ecological Systems

2016- 2017

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Researchers: Lorand Parajdi -"Babes-Bolyai" University, Yves PAPEGAY (INRIA - Sophia Antipolis)

Presentation of the research topic

We had to solve an ecological system. An ecological system is a set of pairs $\{(A_i, \alpha_i)\ 1 \leq I \leq n\}$ where A_i is a point in the plan, α_i is a natural number which corresponds with the number of trajectories (go-return) between that point and a settled-point of departure. At each point of departure M from the plan, we associate the total trajectories T (M) which corresponds with the ecological system:

$$T(M) = \Sigma\alpha_i \times MA_i \ (i = 1, n)$$

To solve an ecological system is to find all the solutions to this ecological system. We had to solve the system for 2 and 3 points.

Brief presentation of the results obtained

By taking particular cases, for the system with 2 points we noticed that the minimal value for T(M) is when the minimal α corresponds to the point which is further from the point M than the other one, so the biggest α should represent the closest point from M. For the system with 3 points we took M as at the intersection of the lines which build with the vertices of the triangle angles of 120 °, we take that as the center of the triangle so that we had T(M) minimal.

Steps in solving the problem

We took for observation some particular cases of 2 points and we noticed here that the minimal value for T(M) is when the minimal α corresponds to the point which is further from the point M than the other one, so the biggest α should represent the closest point from M. We also noticed for the same number of round trips that for two points, we had to put the point M wherever on the segment between the two points.

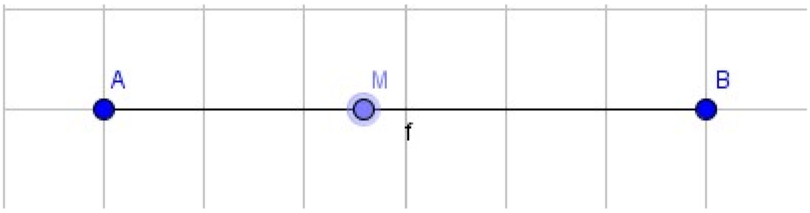

Fig.1 Position of M for the same number of round trips

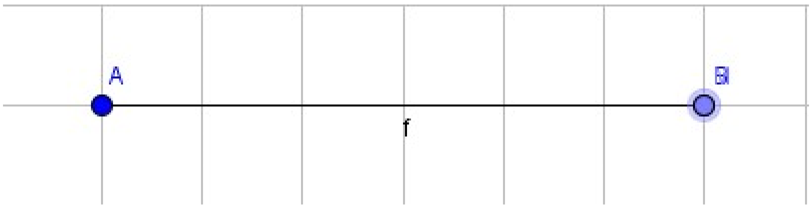

Fig.2 Position of M for different number of return trips

For the case where we had 3 points we put in all types of triangles the point M at the intersection of AM, BM, CM, which constructs angles of 120 ° with the vertices of the triangle. We are still in the process of finding a more general solution, but at the moment we reached this point of the research.

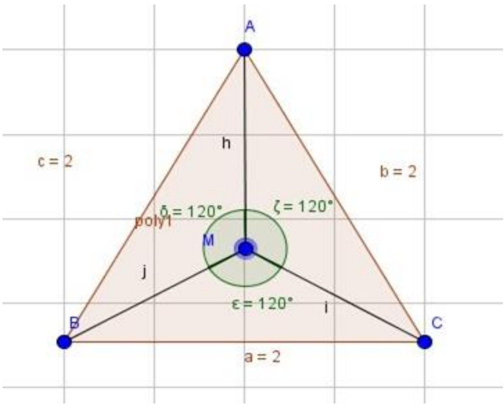

Fig.3 Position of M in the equilateral triangle

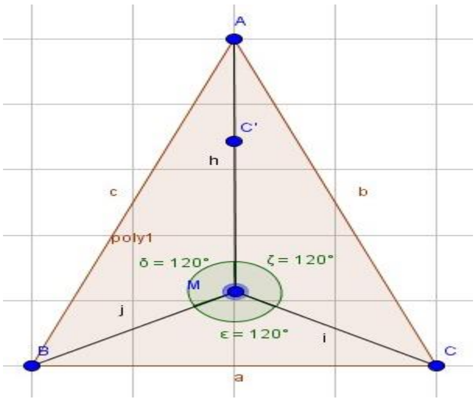

Fig. 4 Position of M in the isosceles triangle

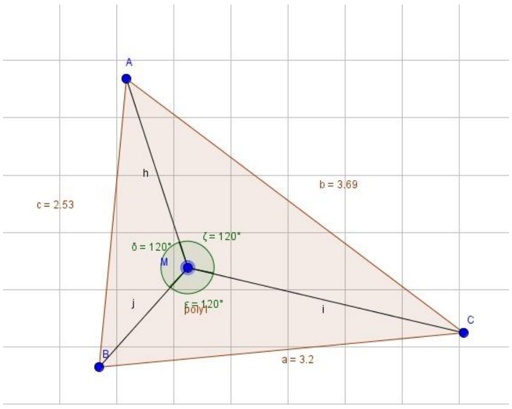

Fig.5 Position of M in any triangle

For a different number of return trips the point M is in the point A for h ≥ i + g, in the point B if i ≥ g + h or in the point C if g ≥ h + i, where h represents the number of return trips from A to M, i represents the number of return trips from B to M and g represents the number of return trips from C to M.

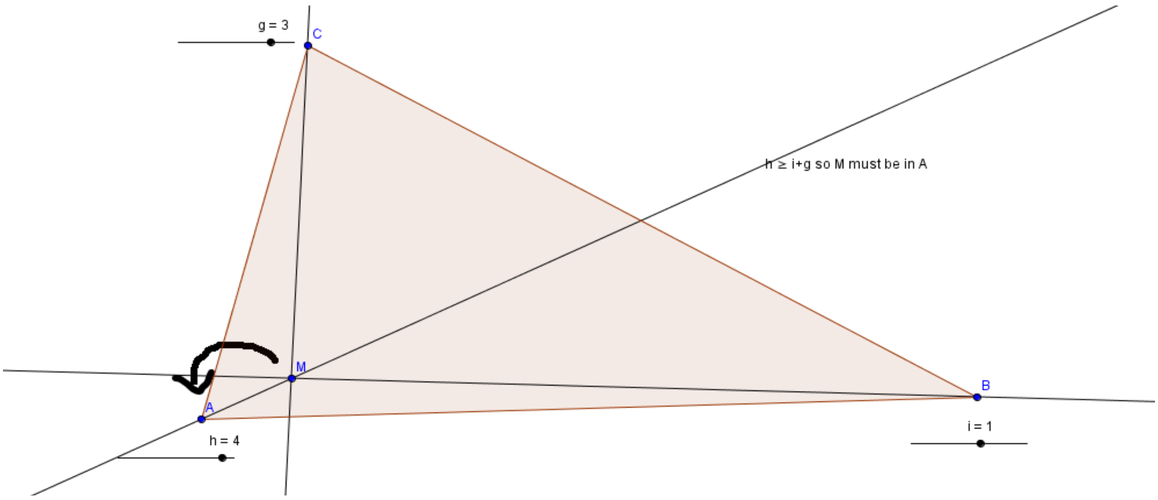

Fig.6 Position of M in any triangle

This article is written by students. It may include omissions and imperfections.

Encryption

2016- 2017

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Presentation of the research topic

The topic of our research was studying the history of encryption and use what we have learned to create our own functional encryption method.

Brief presentation of the conjectures and results obtained

Our research was based on the conjecture that we could combine multiple types of encryption methods and create a new original one. We have managed to do just that and create a functional and unique encryption method.

Our work

Encryption is a branch of mathematics which specializes in securing information and also controlling access in a database. In order to do so either mathematical encryption methods can be used (taking advantage of the difficulty of factorizing large numbers) or quantic encryption methods. Until recently the term encryption referred almost exclusively to the process of converting information (usually a text with a meaning) into an apparently illogical one. Decryption is the process of converting the encrypted back into a text that can be read and understood. A cypher is a pair of algorithms that are capable of both encrypting and decrypting a text. The way it operates is controlled by the algorithm and by the encryption key.

Since the dawn of time, encryption has been used in order to hide certain messages from those who should not be seeing them. The first encrypted document is a written on a clay board and has been discovered in Iraq, and was dated to be from the 16th century B.C. It is a recipe made by a potter which has been encrypted by removing the vowels and modifying the orthography of words. Until the 20th century encryption was only available to human operators that used pen and paper in order to code and decode messages. As a result of the technical progress during the 20th century the encryption methods have become more and more complex thanks to the apparition of computers.

Encryption has been nicknamed a weapon of war, due to its ability to help in battles or even decide the fate of wars. The first uses of encryption in war date to Ancient Greece. The Greeks refuse to obey the Persian king Xerxes and he decided to declare war to Greece. The legend says that in the city of Susa in Persia, lived Demaratos an exiled Greek still loyal to his country. In order to warn the Greeks he wrote a message on a wooden board and then covered it in wax so the message was no longer visible. Herodotus and Gorgo, the wife of Leonidas were able to discover the hidden message. Thanks to this help the Greeks were prepared and defeated Xerxes the day he attacked. This is how stenography was born, an ingenious way of communicating hidden messages that included even tattooing the message on a couriers' body.

Encryption can also be split in two different branches: transposition and substitution. Transposition means that the letters in a message are rearranged in order to form anagrams. This method is mostly used for short messages, but at the same time it is not very safe as the message is very easy to decrypt taking into consideration the limit number of options. The longer the text the safer the transposition is. The most well-known case in which cryptanalysis was used was that of Queen Mary of Scotland. In 1568, Queen Mary flees to England where she is arrested by the order of Queen Elisabeth, under the accusation of having killed her husband. The real reason for her arrest was that she was catholic and considered a treat by Queen Elisabeth. In 1586 she received encrypted letters from her supporters that planned to set her free and remove Elisabeth from the throne. These letters were decrypted and Queen Mary was executed.

During WW2 the situation changes once more, the supremacy being taken by the English with the Colossus encryption machine. The Colossus was considered the precursor of the modern computer and was the base of the technological development in the 20th century. After ARPANet was born in 1969, it was the starting point for developing the World Wide Web. The cryptographist Whitfield Diffie in collaboration with professor Martin Hellman have created asymmetrical encryption, he encryption with a public key. The technique utilizes two keys: one that is public and one that is private. Everybody has access to the public key so they can encrypt a message, but only the owner of the private key can decrypt them.

As Diffie predicted in the 70`s, the era of Informatics began, a post-industrial era where. Due to the size of these transactions it is very important to protect information. For 2000 years encryption was important for kings and queens, for armies and governments, but now it the most valuable product is information. Digital information exchange is an integrated part of society and billions of emails are sent every day. Money can quickly circulate through cyberspace and an estimated half of the worlds` GDP travels everyday trough the network of the International Society of Financial Interbancary Telecommunication allows the development of businesses and even protecting intimacy.

Enigma was the biggest breakthrough in encryption. It was used by the German army since WW1. It had over 17000 encryption keys so even if somebody managed to get hold of an Enigma machine they had no chance of decrypting the message without the correct key.

The Enigma machine also appeared in films and books as the mystery of encrypted messages always fascinated the public.

One of the most appreciated films about the Enigma machine was "The Imitation Game" from 2014 which present the struggle led by Alan Turing to discover a method of decrypting messages sent using it.

Another well know film about Enigma based on real events is "U-571" which is about the efforts to capture the machine by the Americans which secretly board a German submarine.

Our project is based on the complexity of encryption. A simple program of 50 lines of code can encrypt a text in countless ways. The program splits the text in blocks consisting of four letters numbered in alphabetical order (A=1, B=2, …) and then are transformed into a matrix. The matrix is multiplied by a key matrix and then the text is arranged in a linear way. Every number is divide by 31 and the rest of the division is added to 1 and the number is transformed in an alphabetical character.

$$\begin{bmatrix} 3 & 14 \\ 22 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3\times 1 + 14\times 2 & 3\times 2 + 14\times 1 \\ 22\times 1 + 2\times 2 & 22\times 2 + 2\times 1 \end{bmatrix} = \begin{bmatrix} 31 & 20 \\ 26 & 46 \end{bmatrix}$$

$$\begin{bmatrix} 31\%30 & 20\%30 \\ 26\%30 & 46\%30 \end{bmatrix} = \begin{bmatrix} 1 & 20 \\ 26 & 16 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 21 \\ 27 & 17 \end{bmatrix} \Rightarrow \text{B U Q}$$

(%: operator that provides the rest of the division)

We have created a similar program that does the operations in a different order, first it adds 1 and then multiplies the matrixes. The result are 2 identical modes of encryption, but in a different order.

The result of the study regarding the two algorithms has proved that the algorithm cannot be put in another order even if the encryption is still viable, but I will not return the same results.

Conclusions

We have tested if encryption can be associative and after multiple tests we have discovered it is not.

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This article is written by students. It may include omissions and imperfections.

Filling Containers

2016- 2017

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Researchers: Lorand Parajdi, Universitatea "Babes Bolyai", Cluj-Napoca, Yves PAPEGAY (INRIA - Sophia Antipolis)

Presentation of the research topic

Three vases A, B and C have the capacities of 9, 5 and π litres respectively. In the beginning, A has 9 liters of water, B and C are empty. We can pour the water from vase X to vase Y, under the two following conditions:

1) we don't waste the water
2) after pouring the water, X is empty and Y is full.

Prove that for every $\varepsilon > 0$, after a finite number of operations, we can obtain in one of the vases a number of liters of water between $1-\varepsilon$ and $1+\varepsilon$.

Brief presentation of the conjectures and results obtained]

We have managed to solve the original problem and to generalize it using some theoretical results about density in R and the mathematical induction principle.

Our work

1.Theoretical results

A set $X \subset R$ is called dense in R if, for any real numbers $a < b$, there exists $x \in X$ such that $a < x < b$.

Kronecker's density theorem

Let α be an irrational number. The set $A = \{m\alpha+n|m,n \in Z\}$ is dense in R.

Proof: (the idea is taken from the book Matematica de excelenta: pentru concursuri, olimpiade si centrele de excelenta: clasa a IX-a)

Firstly, we should prove:

Dirichlet's Theorem

Let α be an irrational number and p a positive integer. There exist integers m,n with $0 < m < p$ so that $|\alpha m-n| \leq 1/p$. We divide the interval [0,1] in p intervals: [0,1/p], [1/p, 2/p], …, [(p-1)/p, 1]. Let:

$a1 = \{\alpha\} = \alpha -[\alpha]$

$a2 = \{\alpha\} = 2\alpha -[2\alpha]$

...

a(p+1) = {α} = (p+1)α −[(p+1)α]

From Dirichlet's Principle, two of these p+1 numbers will lie in the same interval, let them be a i and a j, with i > j. Then, |xi–xj | = |(i− j)α + ([jα]−[iα])| ≤ 1 p . Taking m = i− j and n = [jα]−[iα], we've proven Dirichlet's Theorem.

Let us use this theorem in proving Kronecker's. Let a', b'∈ R, a'< b' . We should prove that there exist integers m' ,n' such that a < m'α +n'< b. Let d = b' –a'> 0 and take a positive integer p such that 1<d ≤ p (for example p = 1/d +1). From Dirichlet's theorem, there exist integers m,n (m > 0) such that |mα −n| < 1/p ≤ d. As mα − n is irrational, it is not 0. Let mα − n = u. Suppose u > 0 and α > 0 (the other cases are analogue). The numbers u,2u,3u,. . . , ku,. . . are pairwise different and are in ascending order. Than, at least one of them is greater than a'. We prove that the smallest number that exceeds a', let it be ku, is in the interval (a' ,b'). From k's choice, we have that (k − 1)u ≤ a' . If we had ku ≥ b' ,then ku − (k − 1)u ≥ b' –a' = d, that means u ≥ d, that is a contradiction. So, a' ≤ ku ≤ b', that means a' ≤ mkα −nk ≤ b' and mk, −nk are integers. Taking m' = mk, n' = −nk, the proof is complete.

2. Proof

A configuration is a triple (a,b,c), meaning that in the vases there are a, b, respectively c liters of water. We will proceed by induction, proving:

$p(n)$: "We can obtain the configuration $\left(9-5\left\{\frac{n\pi}{5}\right\},5\left\{\frac{n\pi}{5}\right\},0\right)$ from the initial configuration $(9,0,0)$."

Step 1: $p(1)$ is true: $(9,0,0)\to(9-\pi,\pi,0)$.

Step 2: Assume that we have already proven that $p(1),p(2),\dots,p(n)$ are true, for some n. Let's prove that $p(n+1)$ holds. From $p(n)$, we can obtain the configuration $\left(9-5\left\{\frac{n\pi}{5}\right\},5\left\{\frac{n\pi}{5}\right\},0\right)$. As $5\left\{\frac{(n+1)\pi}{5}\right\}=5\left\{\frac{n\pi}{5}+\frac{\pi}{5}\right\}$, there are two cases:

Case I

$5\left\{\frac{n\pi}{5}+\frac{\pi}{5}\right\}=5\left\{\frac{n\pi}{5}\right\}+\pi$

The following operations solve this case:

$\left(9-5\left\{\frac{n\pi}{5}\right\},5\left\{\frac{n\pi}{5}\right\},0\right)\to$

$\left(9-5\left\{\frac{n\pi}{5}\right\}-\pi,5\left\{\frac{n\pi}{5}\right\},\pi\right)\to$

$\left(9-5\left\{\frac{n\pi}{5}\right\}-\pi,5\left\{\frac{n\pi}{5}\right\}+\pi,0\right)$

Case II

$5\left\{\frac{n\pi}{5}+\frac{\pi}{5}\right\}=5\left\{\frac{n\pi}{5}\right\}+\pi-5$

The following operations solve this case:

$\left(9-5\left\{\frac{n\pi}{5}\right\},5\left\{\frac{n\pi}{5}\right\},0\right)\to$

$\left(9-5\left\{\frac{n\pi}{5}\right\}-\pi,5\left\{\frac{n\pi}{5}\right\},\pi\right)\to$

$\left(9-5\left\{\frac{n\pi}{5}\right\}-\pi,5,5\left\{\frac{n\pi}{5}\right\}+\pi-5\right)\to$

$\left(9-5\left\{\frac{n\pi}{5}\right\}+5-\pi,0,5\left\{\frac{n\pi}{5}\right\}+\pi-5\right)\to$

$\left(9-5\left\{\frac{n\pi}{5}\right\}+5-\pi,5\left\{\frac{n\pi}{5}\right\}+\pi-5,0\right)$

By Kronecker's Theorem, the set $A=\left\{5\left\{\frac{n\pi}{5}\right\}|n\in\mathbb{N}^*\right\}$ is dense in the interval $(0,5)$. That means, for every $\varepsilon>0$, there exists $a\in A$ such that $1-\varepsilon<a<1+\varepsilon$ and the problem is solved.

1	3.14159
2	1.28319
13	0.840707
21	0.973444
99	1.01767
212	1.01761
664	1.01746
1568	1.01715
2133	1.01685
5523	1.01563
12416	1.01318
16936	1.01074
26089	1.00586
52192	0.996094

fig 1: The closest result obtained for the first 60000 numbers

3. Generalization

Let the vases have their capacities (a,b, c). We are wondering if there exists a real number x such that for every $\varepsilon > 0$, we can obtain x' liters of water in a vase, such that $|x-x'| < \varepsilon$. Dividing by a and after a change in notation, the capacities of the vases become (1,a,b).

If both a and b are rational, the conclusions is not true.

If one of a and b and be is rational and one irrational, in some cases the desired result can be attained, but in a general case a weaker hypothesis is needed (if, for example, we could waste water and we had an infinite amount of water, the desired result can be obtained).

If both a and b are irrational, the problem gets much harder

Conclusions

Using the strong results provided by Kronecker's theorem, we managed to show that we can construct a sequence that converges to 1. Because the generalization is quite difficult, it remains an open problem at this given time.

This article is written by students. It may include omissions and imperfections.

How to Kill the Hydra

2016-2017

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The research topic
Is there any strategy that allows Heracles to kill the hydra?

Definitions
• Maximum node (a): every head is a *maximum node*;
• Value of a node: a node at height *h* has value *v* if it is connected to *v maximum nodes* at height *h+1*;
• Maximum node (b): a non-head node at height *h* is a *maximum node* if it has the highest value among all of the nodes at height *h*.
• Vector: we associate to every hydra of height *n* a vector with *n* components. The i-th component of this vector is the maximum among the values of the knots at height n-i.
• Order relation between vectors: a vector is smaller than another vector if it has less components or the first different component they have is smaller. Formally, $(a_1,\dots,a_n) < (b_1,\dots,b_m)$ if $n < m$, or $n=m$ and there exists an index i such that $a_i < b_i$ and $a_j = b_j$ for every $j< i$.

Procedure

Beginning from the root of the hydra, following a path of maximum knots, up the maximum height, we cut the head we reach. Repeating this procedure, we will be able to defeat the hydra. We prove that, at each move, the vector associated to any hydra decreases after each move. In particular the first component from the right greater than 1 will decrease by 1, or, if every component is equal to 1, it will get shorter.

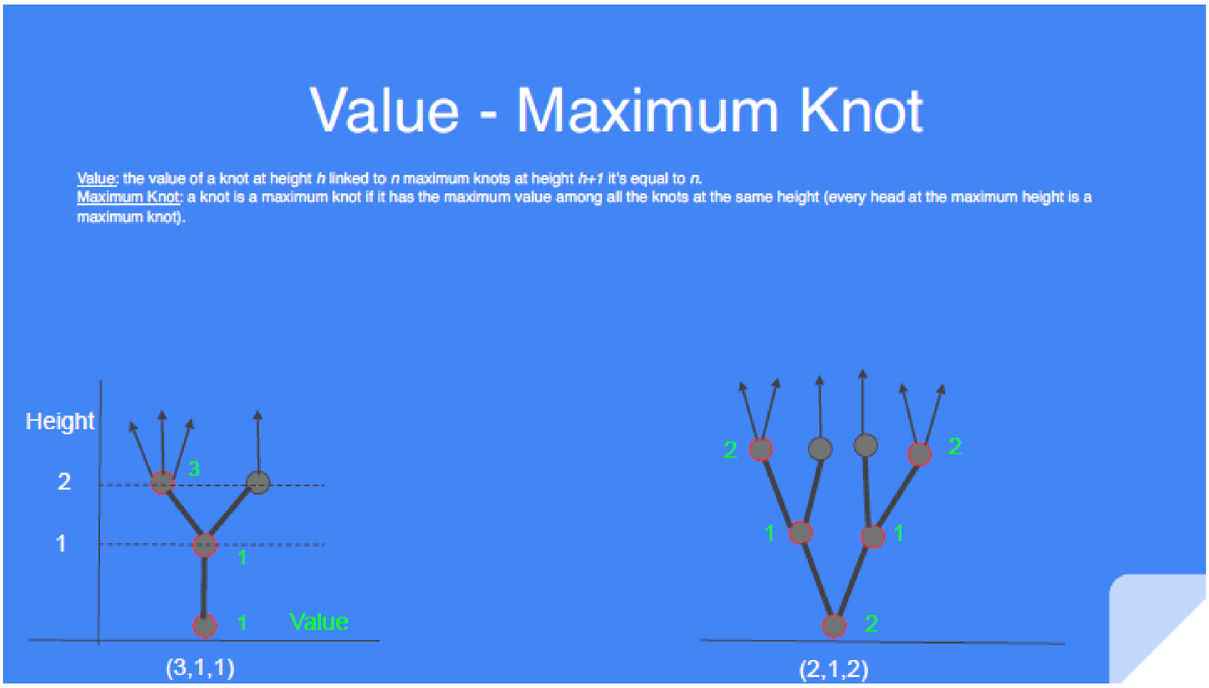
If the vector has every component equal to 1, we cannot have 2 heads at the maximum height (otherwise, following a path of maximum knots downwards, as soon as the two paths connect, that knot will have value 2). Therefore, cutting the only head at the maximum height, the vector will shorten up by 1.

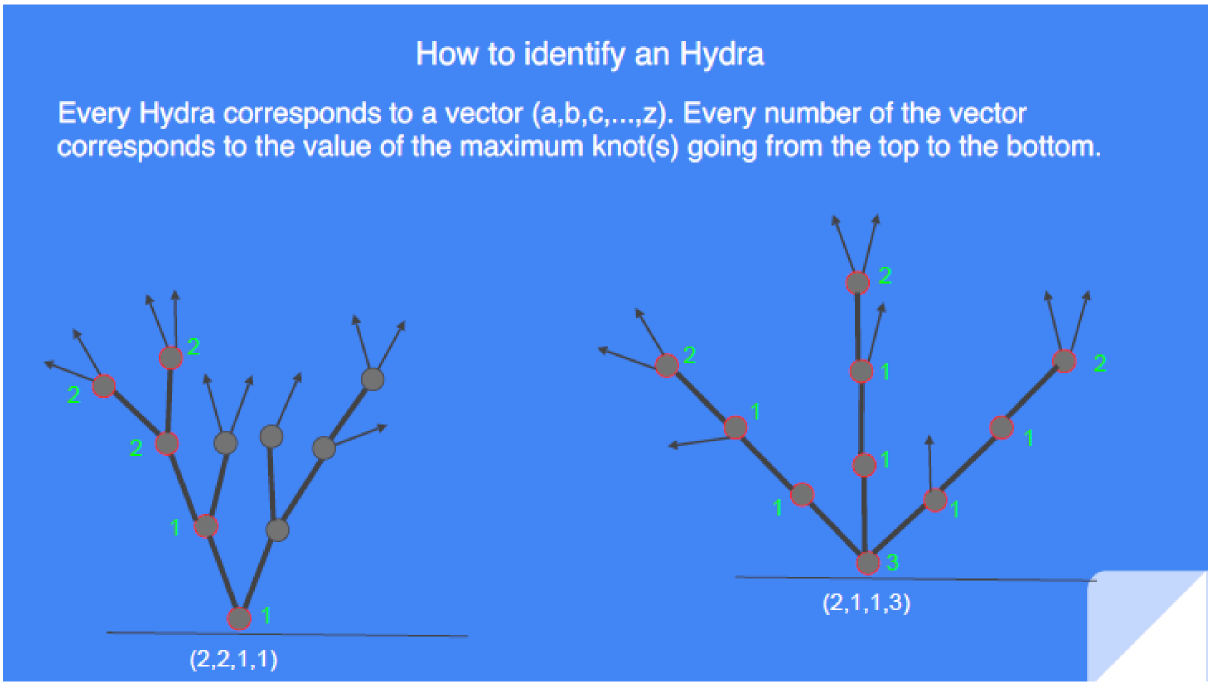
Let's evaluate how the vector changes after the cut, if it has some (or all) components different from 1. It's important to observe that if the vector has k components from the right equal to 1, the maximum knot is not unique starting from height k+1. The number of components equal to 1 from the right is k ($k \geq 0$). When we cut the head we previously mentioned, the value of the knot right beneath it will decrease by 1 and, having other maxima at that height, this knot stops being a maximum. Likewise, the value of the knot right beneath this knot will decrease by 1 and it won't be a maximum anymore. This pattern continues up to the maximum knot at height k, which, being the only one, it will still be it (in some cases, with others). We can now observe that the values of the knots at a height between $n-1$ and $k+1$ are unchanged and the knot at height k decreased by 1.

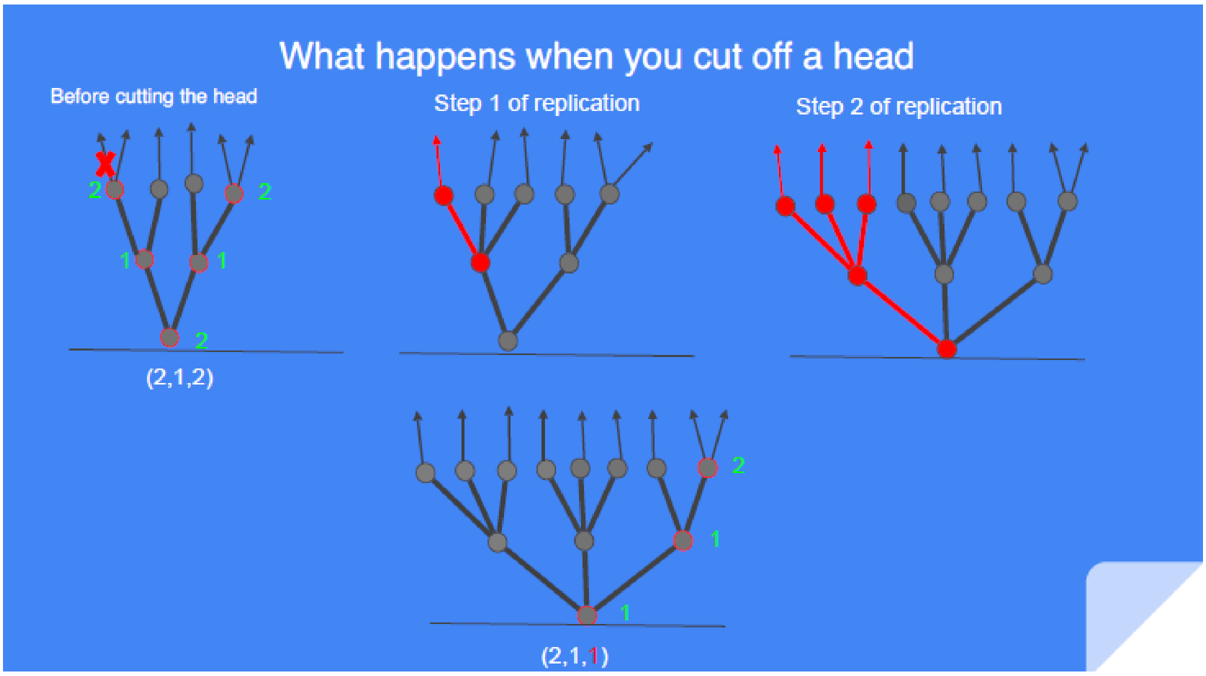
Therefore, we just need to show that these values remain unchanged after the replication.
If the replication stimulus comes from a knot which is not a maximum, the knot's value right beneath it doesn't change and doesn't become a maximum. Hence neither do all the knots underneath. Also, in every replication, the maximum value at heights that are greater than the replicating knot remain unchanged, since the knots that are generated are copies of pre-existing knots at that height. Let's notice then that the replication in this case leaves the first n-k components of the vector unvaried, since up to k the replication stimulus proceeds in non-maximum knots (that can't become such during the replication).

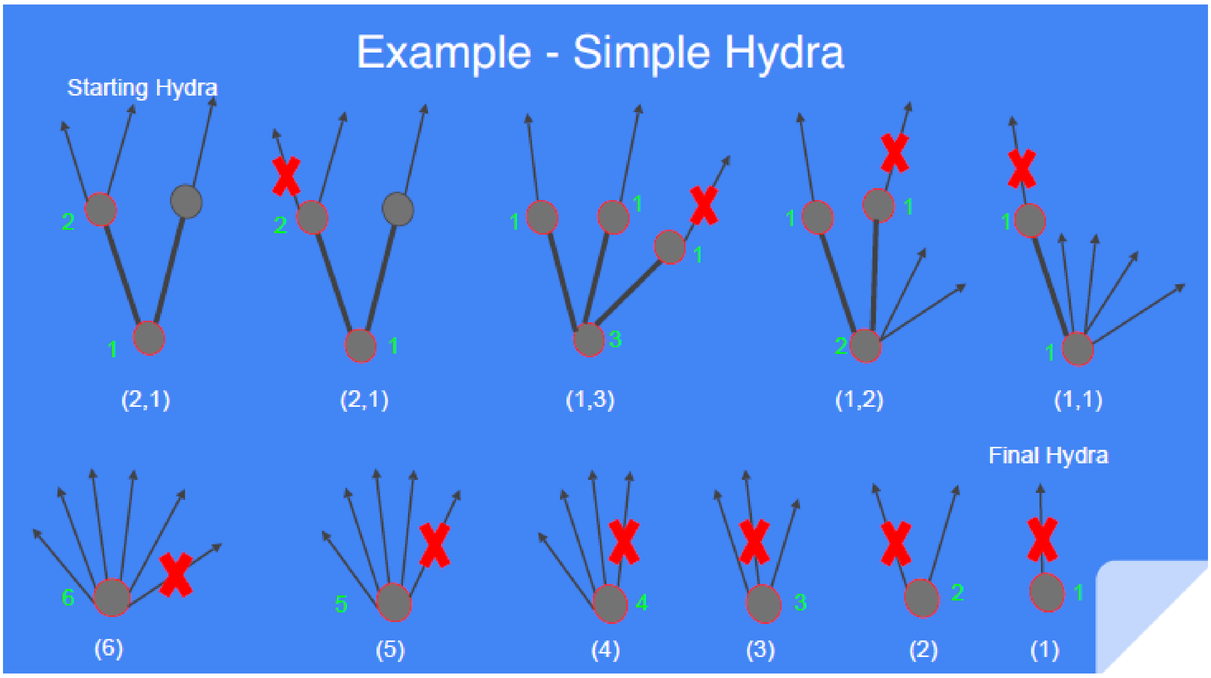
Lastly, let's prove through induction that an infinite decreasing sequence of vectors doesn't exist, so a hydra can't survive an infinite amount of attacks.
P(n) = "An infinite decreasing sequence of vectors with *n* components does not exist".
Basis: n=1. An infinite decreasing sequence of positive integers does not exist (well ordering principle).

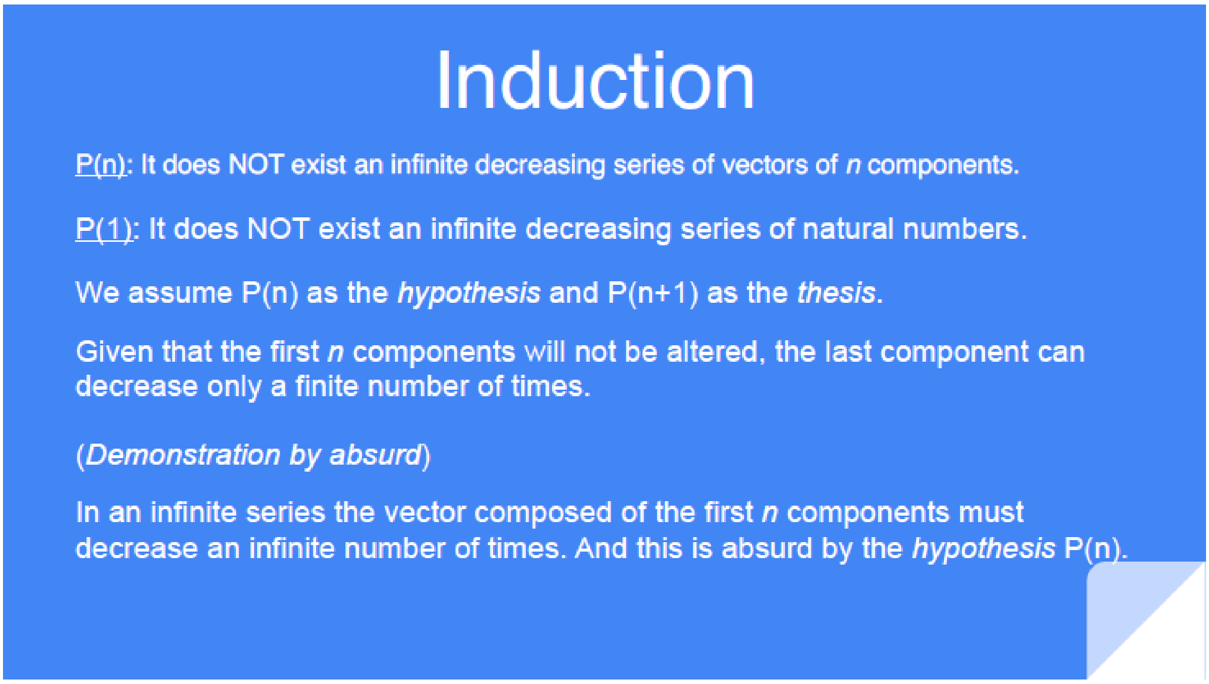
Inductive hypothesis: Let us assume P(n) holds for some value *a* (P(a)).
Inductive step: Given that P(a) holds, let us prove that P(a+1) holds for any positive integer *a*. Leaving the first *a* components unchanged, the last one can only decrease for a finite amount of times. Therefore, in an infinite sequence, the vector composed of the first *a* components must decrease an infinite amount of times, contradicting the inductive hypothesis. Q.E.D.


Value - Maximum Knot
Value: the value of a knot at height h linked to n maximum knots at height h+1 it's equal to n.
Maximum Knot: a knot is a maximum knot if it has the maximum value among all the knots at the same height (every head at the maximum height is a maximum knot).
Height
2
1
3
1
1
Value
(3,1,1)
2
2
1
1
2
(2,1,2)


How to identify an Hydra
Every Hydra corresponds to a vector (a,b,c,...,z). Every number of the vector corresponds to the value of the maximum knot(s) going from the top to the bottom.
2
2
2
1
1
(2,2,1,1)
2
2
1
2
1
1
1
1
1
3
(2,1,1,3)


What happens when you cut off a head
Before cutting the head
Step 1 of replication
Step 2 of replication
(2,1,2)
(2,1,1)


Example - Simple Hydra
Starting Hydra
(2,1)
(2,1)
(1,3)
(1,2)
(1,1)
Final Hydra
(6)
(5)
(4)
(3)
(2)
(1)


Induction
P(n): It does NOT exist an infinite decreasing series of vectors of *n* components.
P(1): It does NOT exist an infinite decreasing series of natural numbers.
We assume P(n) as the *hypothesis* and P(n+1) as the *thesis*.
Given that the first *n* components will not be altered, the last component can decrease only a finite number of times.
(*Demonstration by absurd*)
In an infinite series the vector composed of the first *n* components must decrease an infinite number of times. And this is absurd by the *hypothesis* P(n).

Just Untold Things

2016-2017

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A. The case that forces the virtual shrink of the land

UNTOLD FESTIVAL attracts every year a very large number of people, from inside and outside the country.

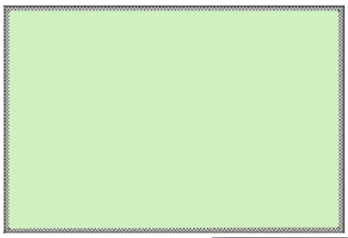
Apart from the hotels, the Council of Cluj-Napoca provides the spectators a land of 100x150 square meters where they can deploy rented tents, with a square shaped base.

I.The spectators rented 957 tents and demanded their deployment on the same land with a round scene with a 2 meter diameter for karaoke

The Math en jeans team studied that is possible to deploy all of the 957 tents and there will also be room for the karaoke scene.

1.To make sure that the scene and the tents do not cross the limits, we will decrease the land with an imaginary 1 meter frame where we can’t set the gravity center of the studied objects (the circle with a 1 m radius and a square with a 2 m side)

This is the available surface now $(150-2)(100-2)=148\cdot 98=14504\ m^2$

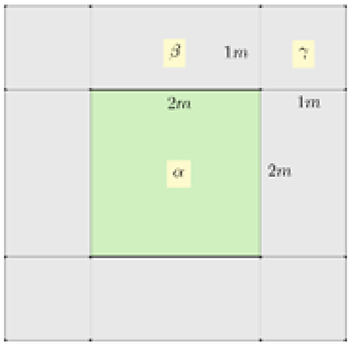

2.To avoid the imbrication of the objects, we set a 1m frame for every tent,so the surface of every tent will be $4 \cdot 4 = 16\ m^2$

In this conditions, the 957 tents with bigger surfaces will occupy $957 \cdot 16 = 15312\ m^2$

If 15312 > 14504, we can say that on the land there will be no space for the tents and the karaoke scene.

However, from the calculations we can see that the 957 tents have enough space

$957 \cdot 2^2 = 957 \cdot 4 = 3.828$ and $\frac{14504}{3828} \approx 3,7$, so this strategy need to be forgotten because it leads us to a contradiction.



3. We choose another strategy, expanding the surface of the tents: we set an imaginary margin of 1 meter, with round corners

Observation: the four corners are the quarters of a 1 m radius circle

The new surface of every tent is now

$\alpha + 4(\beta + \gamma) = 4 + 4\left(2 \cdot 1 + \frac{\pi \cdot 1^2}{4}\right) = 4 + 8 + \pi = 12 + \pi \cong 15{,}14.$

The bigger surface of the 957 is 957*15,14 = 14.488,98≅ 14.489 m^2

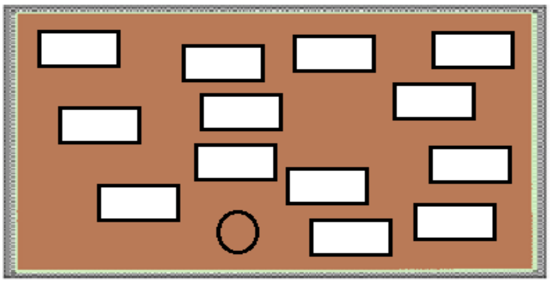
We compare the available surface with the expanded surface of the tents

Considering that the surface for the karaoke scene is $\pi \cdot 1^2 = 3{,}14\ m^2$ and 14504 – 14489 = 15m^2, the conclusion is that we can set on the available land the 957 tents and also the karaoke scene.

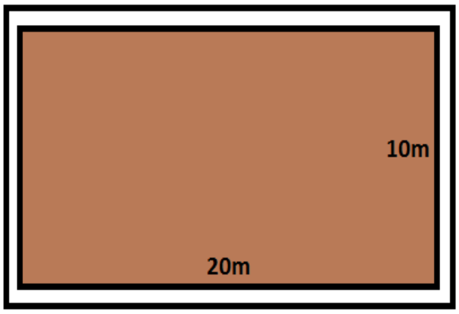
II. The Untold Festival is about to begin. Until then there are more organizational problems that need to be solved.

The Math en Jeans team wishes to help the designers that have to set, on an 15000 square meters land,55 stages where the best artists will perform, and also a circle shaped bar with a 1m radius.

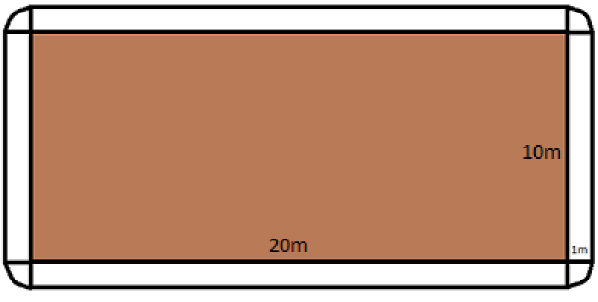
We studied that, wherever the artists will want to set the stages, there will be space for the bar.



1.To make sure that the bar does not beat the limits of the land, we decrease its surface with a 1m imaginary frame (we can't set the bar's gravity centre on it)
The available surface is now
2.To avoid the superposing of the objects on the land, we set a 1m imaginary frame for each stage, so the new surface will now be 22*12=264 m^2
In these conditions, the 55 stages will occupy 55*264=14520 m^2.
Knowing that 14520>14504, we can say that on the land we can't set the stages and we exclude the possibility of having a bar there. ☹
Still, from the calculations results that the 55 stages have enough space:55*20*10=1100m^2 and $\frac{14504}{1100}$ =13,18, so the strategy needs to be abandoned, because it leads us to a contradiction.



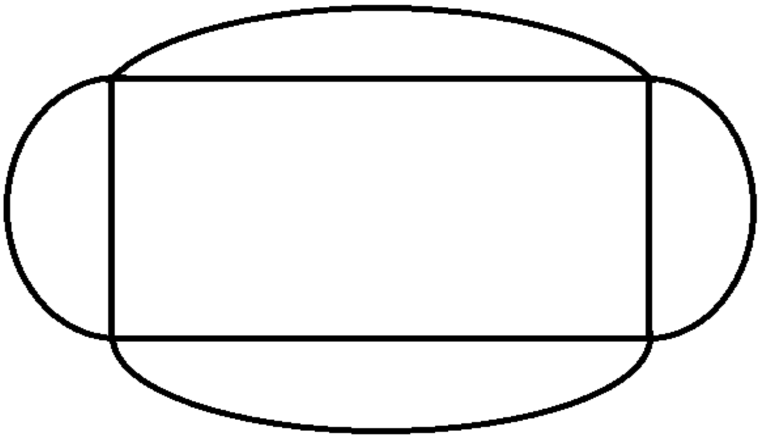
3.We choose another strategy, of expanding the surface of the stages: we set an imaginary margin of 1m, with round corners
Obs.: The 4 rounded corners are the quarters of a circle with a 1m radius
The expanded surface of one stage is now 20*10+2*20+2*10+3,14=263,14 m^2.
The surface of 55 stages is 55*263,14=14472,7 m^2
We compare the available surface with the expanded surface of the 55 stages
14504-14472,7=31,3 m^2



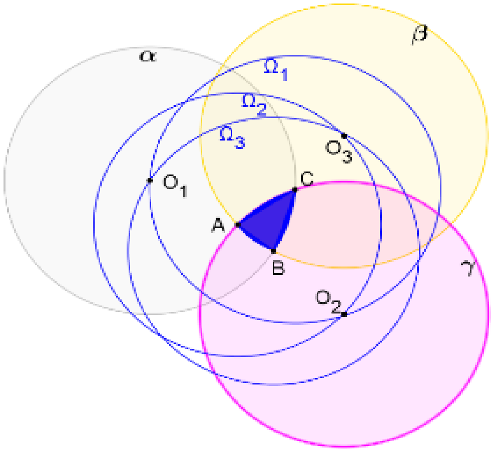
Conclusion

We can set on the available land the 55 stages and also the bar.

For the next Maths en Jeans congress we proposed to find a strategy of expanding the surface for the following shape:

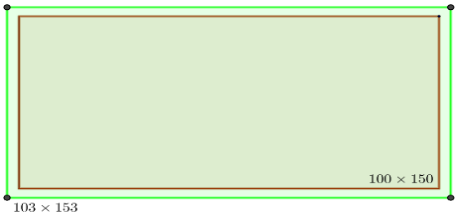


B.The case that forces the virtual expansion of the land
I.After the ending of the Untold festival, the council of Cluj-Napoca has organised on the same land of 100x150 square meters an outdoor disco that has a spotlight which scans the surface with a light circle with a 3 meters radius. If on the land are, at a certain moment, 1116 young people, demonstrate that, ignoring their position, the spotlight can light at least one group of 3 young people.
Solution:
We consider that the 1116 persons are 1116 points. We draw around every point imaginary discs with a 3-meter radius. We have to show that there are at least 3 discs intersect each other.
The virtual discs are α, β,γ they have a 3m radius and their centres, are one the ABC surface (a non-empty amount)
All of the points situated on the ABC surface (blue surface) can be the centres of some 3m radius circles which include the points.



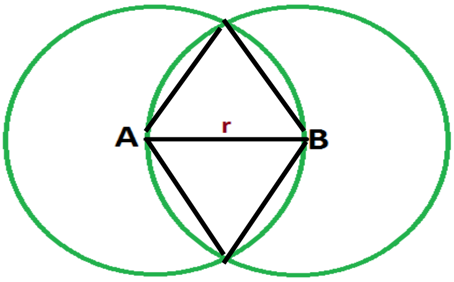
1. We calculate the surface of the 1116 discs $S_1 = 1116 \cdot \pi \cdot 3^2 = 31538{,}16$ square meters.

2. To make sure that a person can stand on one of the margins of the land, we make an imaginary expansion with a m border.
We calculate the surface of the expanded land (the initial land plus the expansion)
$S_2 = (100+3)(150+3) = 103 \cdot 153 = 15759$ square meters
3. We calculate the proportion of the surfaces
$$\frac{S_1}{S_2} = \frac{31538{,}16}{15759} \approx 2{,}01 > 2$$
Conclusion:
The proportion of the 2 surfaces is bigger than 2, which means that there are portions where 3 discs are superposed(the generalized principle of the box),so the spotlight can cover 3 young people at once.



Generalization: We assume that in 2D space we have to set a minimum number of points with the property that, if they are placed random on a land, there are at least 3 points in a circle with a 3m radius
-We associate for each point an imaginary disc of 3m radius, the surface of n discs being 31538,16
-We expand the dimension of the land with a 1.5m imaginary band;
-The existence condition of a circle which will include 3 points, regardless of their position in space
So the number is obtained using the following relation:
-If we have to establish a number of points with the property that they are placed randomly on a land with the dimensions 150x100m, there are at least 3 points that can be included in a circle with a 3m radius, the condition will be that the proportion of the surfaces is bigger than 3 (<3m).
C.
II. We are n students that decided to stick together, and not to depart 10 m from each other.
We will demonstrate that, anytime during our walk, our group can be surrounded by an imaginary circle with a $5\sqrt{3}$m radius.
We choose 2 persons (A and B), that are situated at the maximum distance. We draw the circle with the centre in A and each person that has to be inside the circle, because if we will presume that a person is outside it, the distance between him and the centre will beat the distance AB, which is the maximum.
In the same way, we draw a circle with the centre in B.
So the persons from our group have to be at the intersection of the circles. We note the intersection points C and D, and the shape determined by the points A, B, C, D is a rhomb with the leg d.
If we set O the middle of AB, it results that CO is the height in an equilateral triangle.
CO=$\frac{d\sqrt{3}}{2} \le 5\sqrt{3}$m.

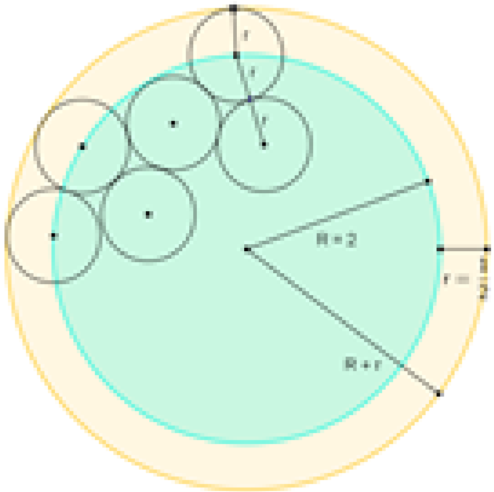
The circle with the centre in O and with a $5\sqrt{3}$m radius contains all of the persons in the group ($5\sqrt{3}\simeq 8{,}66$m).



For example: At Untold, where there will b n spectators, at the distance d between each other, we can ensure the security, all of the persons need to be in a circle with a $d\sqrt{3}$m radius.

D. The case of determination of the relative density

At rock concerts, the spotlight special effects are very appreciated by the public. These include many colourful leds included in a glass sphere that light rhythmically. Prove that in a 2m radius glass sphere there are 2017 steady points (ex: leds) or mobile points (ex: mosquitos, fire beetles),and there will always be 17 points included in a sphere with the radius



Solution:
If in 2D space we had to find a way to compare surfaces, in 3D space we have to compare volumes. If around each point we build imaginary spheres with the r radius, the condition that in each moment there are 17 points which can be included in a r radius sphere is that these spheres do not superpose each other (between 2 spheres with a r radius there has to be a distance d>0 so that if the d=0, the spheres are tangent).

1.We create 2017 imaginary spheres with the centre in the 2017 points and we write down their volume; $V_1 = 2017\frac{4\pi}{3}(\frac{1}{2})^3$.

2.We calculate the volume of a bigger sphere, having a $R + r = 2 + \frac{1}{2} = \frac{5}{2}$ radius

$$V_2 = \frac{4\pi}{3}\left(\frac{5}{2}\right)^3$$

3.We calculate the proportion

$$\frac{V_1}{V_2} = \frac{\frac{4\pi}{3}\cdot 2017\cdot\left(\frac{1}{2}\right)^3}{\frac{4\pi}{3}\cdot\left(\frac{5}{2}\right)^3} = \frac{2017\cdot\left(\frac{1}{2}\right)^3}{\left(\frac{5}{2}\right)^3} = \frac{2017}{2^3}\cdot\frac{2^3}{125} = \frac{2017}{125} \approx 16{,}1$$

Conclusion

There always are 17 points that can be included in a $\frac{1}{2}$ m radius sphere

Generalization

Assuming that a 2m sphere contains points that move chaotically, prove that there always are points that can be included in a smaller volume (a $\frac{1}{2}$ m sphere) if and only if $\left(\frac{r}{R+r}\right)^3 > \frac{k}{n}$.

We act just like in the particular case presented above

-We find the total volume of the imaginary spheres $V = \frac{4\pi}{3}(R + r)^3$

-We find the volume of and imaginary sphere with the radius $R + r$

-We calculate the proportion

$$\frac{nv}{V} = \frac{n\cdot\frac{4\pi}{3}\cdot r^3}{\frac{4\pi}{3}(R+r)^3} = \frac{nr^3}{(R+r)^3} = n\left(\frac{r}{R+r}\right)^3$$

-If $\frac{nv}{V} > k \rightarrow n\left(\frac{r}{r+R}\right)^3 > k \rightarrow \left(\frac{r}{r+R}\right)^3 > \frac{k}{n}$.

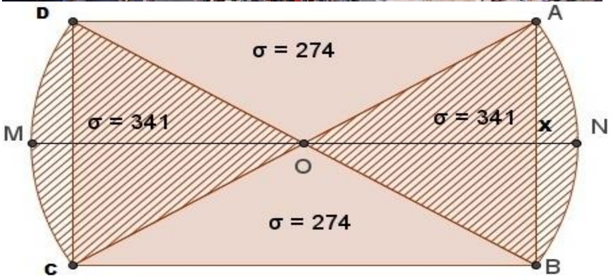
This kind of problems are used to find the optimal density of living organisms (establishing the optimal number of bee families in an acacia forest, calculating the optimal habitat for keeping a special base on the Moon or on Mars alive etc.) or of objects (the location of gas stations, of traffic lights, of windmills, the positioning of the exhibits in a museum etc.), taking in consideration the necessary vital space and/or social norms or technical norms

E. Trevi Fountain – The coin problem

It was found that at the end of each day, the tourists throw in Trevi Fountain around 3,000 euro. We need to prove that the cleaning crews, who donate the money, will find a place within a radius of 0.5 m where there are at least 2 euro.

We only know that Trevi Fountain has the dimensions of 26.3 m × 49.15 m.

In order to solve the problem, we need to draw the diagonals BD and AC, noting their intersection with an O.
After measuring the ODA and AOB angles, we find that their approximate sizes are 120 degrees and 60 degrees, respectively.
In the AOX right-angled triangle, we use the formula of the cosine of 30 degrees to find the hypotenuse OA = 24.6 m.
The AB segment represents the radius of the circle sector OAB, whose area is about 341 m2.
The next step we need to take is finding the XN segment, which is equal to 2.5 meters, according to the formula of the circle sector's area.



Thus, we can calculate the area of ODA equal to the multiplication of (AB and AD) over 2, which is approximately 274 m^2.
Following the previous calculations, we can say that the Trevi Fountain occupies an area of approximately 1230 m^2.
The last phase is to draw around each 1 euro coin a virtual circle which has a radius of 0.5 m.
Since the coins can be very close to the edge of the fountain, we need to add a band of 0.5 m that surrounds the fountain, noting with S the sum of the virtual areas of the 3000 Coins and S ', the fountain area bordered band of 0.5 m width.
In general, if the S over S '= m, we can conclude that there will be m + 1 coins located inside a circle with the radius of 0.5 meters.

Thus, $S = 1334\ m^2$;
$S = 3000\ pi\ (0.5)^2 = 2,355\ m^2$.
$m = 2355/1334 = 1.76$.
$m > 1$, which means that certainly the cleaning crew will find a circle 0.5 meters which contains 2 one euro coins.
Personally, I had a hard time while solving this problem when I had to calculate the area of the fountain with a formula that gives a value close to that on the internet (and in reality, obviously) , which was found based on topographic measurements.
The value ratio of 1.76 suggests a new conclusion, namely: perhaps if we increase the radius to 0.6, instead of 0.5, we will find a place where there are 3 coins of 1 euro, instead of 2.

Lost at Sea

2016- 2017

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Presentation of the research topic

We are lost at sea during a storm at exactly 10 km straight from the coast. What is the shortest way the boat should follow in order to reach the coast if we do not know in which direction it is.

Brief presentation of the conjectures and results obtained

We obtained the shortest distance of 67.73 km by drawing a circle with a 10km radius and a n-edges-polygon that is circumscribed to the circle. The boat, where we are on, is the centre of the circle. The formula we used is D=2*(n-1)*R*tan($\frac{180°}{n}$)+R.

The text of the article

<u>First approach</u>

One of our first ideas was to make a circle with a radius of 10 km and our boat is the centre of the circle. As we thought the coast would be a line tangent to the circle, so the easiest way was to simply go on the radius and then on the circle until we reach the coast (line). In other words, the perimeter of the circle plus the radius: D=2*π*R +R=2*π*10+10≈ 72.8km (see Figure 1). In order to find a shorter distance, we improved our formula by changing the perimeter of the circle with the one of a polygon with n edges which is circumscribed to the circle. In addition, one of the edges of the polygon is the coast (line) and the new formula is D=(n-1)*l+R, where “n” is the number of the edges of the polygon and “l” is the length of the circle.

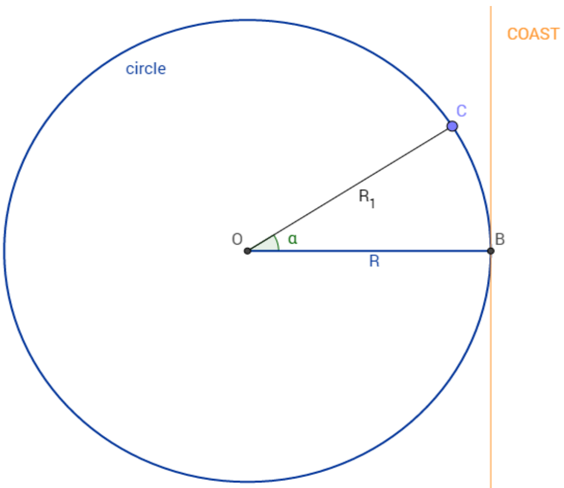


Figure 1. The circle and the coast

Now we have to find a rule between the “n” and “l”. We calculate “l” by using the angle α in the right triangle formed by 2 radiuses and the coast: l=2*R*tanα (1) (see Figure 2). We can also calculate the angle α by using “n”, because the polygon is circumscribed to the circle: $\alpha=\frac{360°}{2*n}=\frac{180°}{n}$(2). Replacing in the initial equation with (1) and (2), we obtain our final equation: D=2*(n-1)*R*tan($\frac{180°}{n}$).

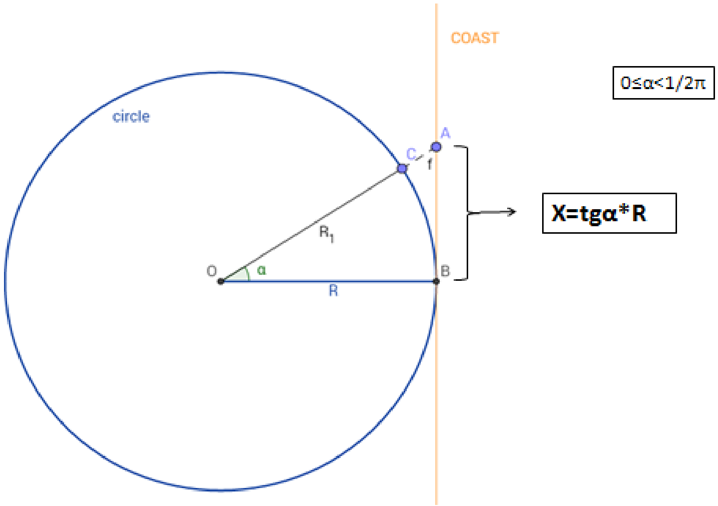
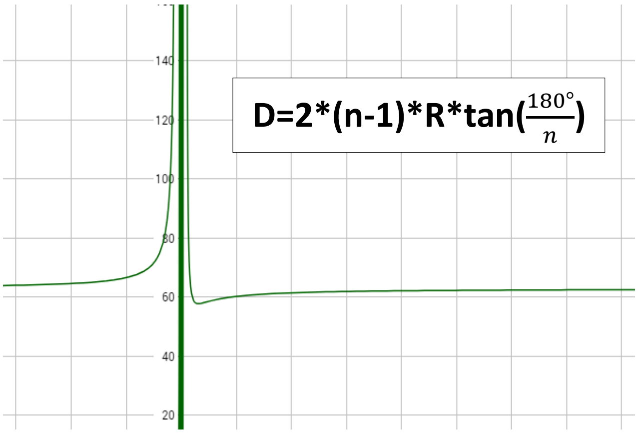
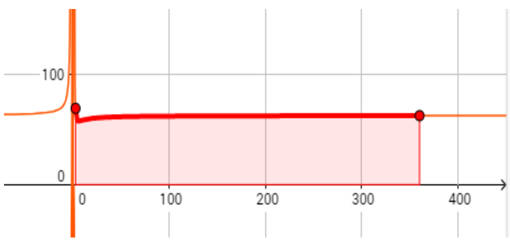


Figure 2. Calculating “l” using the angle α

We made a graph to the function (see Graph 1). We can see that this is an increasing function and we only need the part of it when n≥3 (see Graph 2). Because we need just natural numbers, we also made a table with some natural values “n” from n=3 to n=360 and the distances obtained to see when we reach the smallest distance (see Table 1).



Graph 1. Representation of the distance function



Graph 2. Representation of the distance function for n=3 to n=360

n-edges	α	α-rad	tanα	L- edge lenght	Distance
3	60	1,047198	1,732050808	34,64101615	**79,2820323**
4	45	0,785398	1	20	**70**
5	36	0,628319	0,726542528	14,53085056	**68,12340224**
6	30	0,523599	0,577350269	11,54700538	67,73502692
8	22,5	0,392699	0,414213562	8,284271247	**67,98989873**
9	20	0,349066	0,363970234	7,279404685	**68,23523748**
10	18	0,314159	0,324919696	6,498393925	**68,48554532**
12	15	0,261799	0,267949192	5,358983849	**68,94882233**
15	12	0,20944	0,212556562	4,251131233	**69,51583727**
18	10	0,174533	0,176326981	3,526539614	**69,95117344**
20	9	0,15708	0,15838444	3,167688806	**70,18608732**
30	6	0,10472	0,105104235	2,102084705	**70,96045645**
36	5	0,087266	0,087488664	1,749773271	**71,24206447**
45	4	0,069813	0,069926812	1,398536239	**71,53559451**
60	3	0,05236	0,052407779	1,048155586	**71,84117955**
90	2	0,034907	0,034920769	0,69841539	**72,1589697**
180	1	0,017453	0,017455065	0,349101299	**72,48913244**
360	0,5	0,008727	0,008726868	0,174537356	**72,65891074**

Table 1. Values of the distance depending on ''n''

We can observe from the table that the smallest distance is reached when n=6, which means that the polygon is a hexagon (see Figure 3).

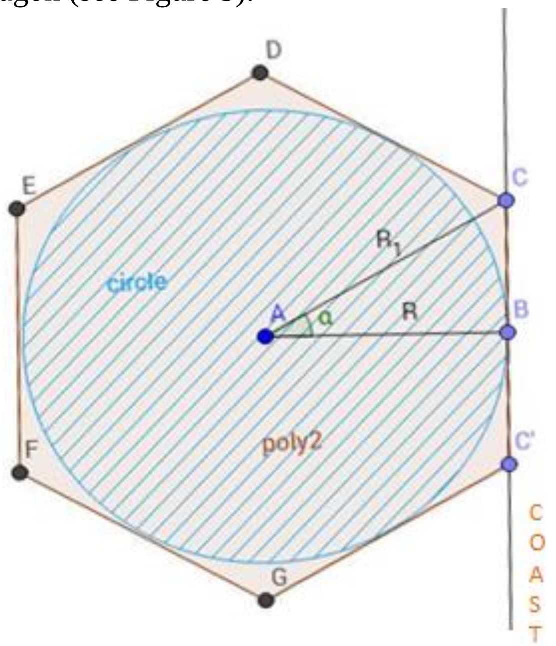


Figure 3. The way that the boat should follow (a hexagon)

Second approach:

We do 10 Km in a random direction. Then we draw a perfect circle. (see Figure 4)
Like this we are sure to find the coast but in the worst case we do 72,83 Km.
This time we are not doing the complete circle but only the ¾ (see Figure 5). And when we finish to do them we have to make 10km in parallel of the first 10km (2 x π x radius) as we did before. (see Figure 6)

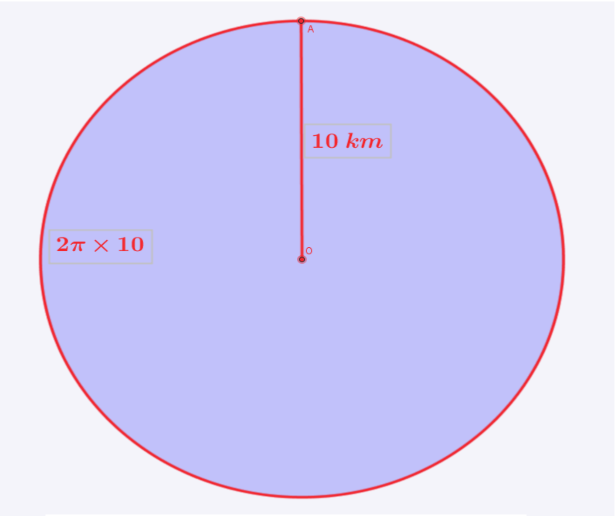


Figure 4. The circle

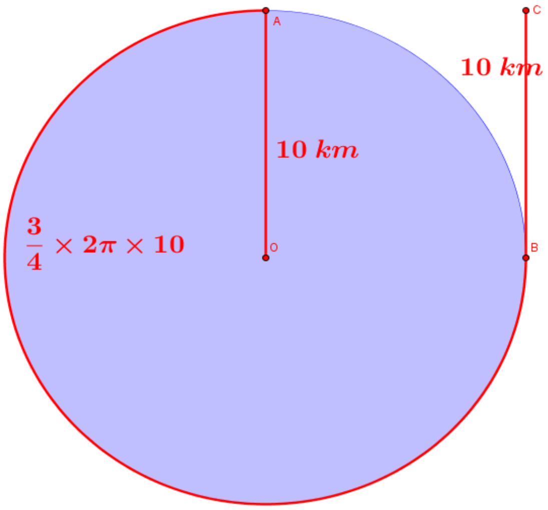


Figure 5. ¾ of the circle

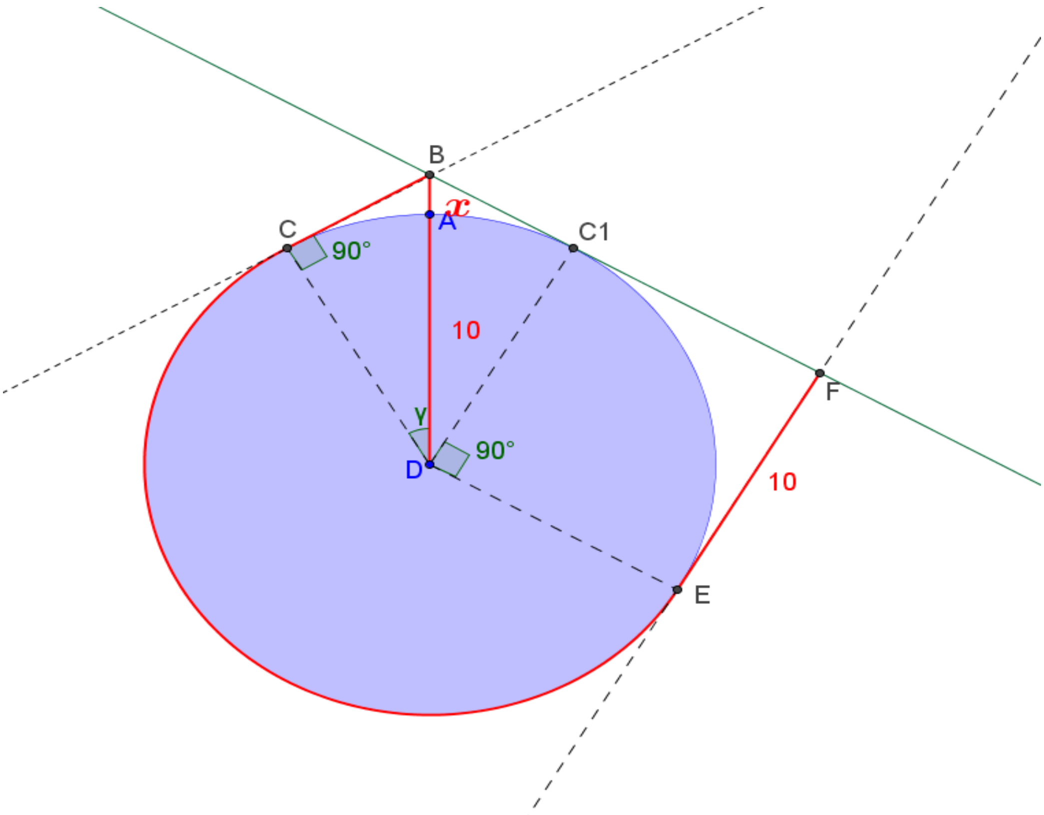


Figure 6. Final route

Conclusions

First approach

It was a difficult road to achieve those final results and even so, we cannot say we didn't enjoy it. At first, we had a hard time to see where to start from. But once we knew this, it became the turning point for our research. Every problem has a lot of ways to solve it, some being better methods than others. But isn't that what a research is? To find different ways to reach results for the same problem. So, our final result to reach the coast on the shortest distance is 67.73 km.

Second approach

For the finish, when we arrive at the circle, we do 10km, more « x ».
Like this, we can do less than ¾ of the circle.

$$10 + x + \sqrt{x^2 + 20x} + \left(\frac{3}{2}\pi - 2\mathrm{arccos}\left(\frac{10}{x + 10}\right)\right)\times 10 + 10$$

Like this we are sure to find the coast but in the worst case we do 63,97 Km.

This article is written by students. It may include omissions and imperfections.

Magic or Not

2016-2017

By Chereji Iulia, Stan Ioana, Varga Andrea, grade 9

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Teachers: Sandor Nicoleta, Marciuc Daly

Researcher: Yves Papegay

Our research topic

We would like to determinate the probability of having a favourable case when combining 24 cards in all the possible ways. We say that a case is *favourable* if, starting from on arbitrary card in a group of 6 successive cards, following a special path, we arrive at the same card in the group of the initial 6. By *special path* we understand continuously taking the same number of steps as the number of the card we have come to. For example, if we arrive at card number 3, we take further 3 steps.

Brief presentation of the conjectures and results obtained

The case of combining a list of different numbers is rather well known, but in our problem, some of the combinations overlapped, so we needed a special type of combining them. After successfully determining the number of possible cases in a mathematical way, we attempted to similarly find the number of favourable cases as well. Here, however, we got stuck. So we decided to generate all the possible cases using CodeBlocks and create an algorithm to test them. In the end, we obtained an approximate value of the probability.

1. Generating all the combinations

To generate all the combinations, without repeating them, we used the algorithm in ***Figure 1*** and ***Figure 2***.

Generally, the lexicographic consequent of each combination of the 24 cards can be determined using 3 steps:

1. We search for the biggest index, j, so the number on the card corresponding to j can be increased.
2. That number is increased by the smallest satisfactory amount.
3. We search the fastest way to extend lexicographically the new combination in order to fill in the model.

We inserted the numbers on the cards in a vector and noted n the number of cards in use; in the general case: n=24.

We go through the vector that contains the numbers on the cards, starting from the index j=n-1 and continuing to decrease the index j by 1, until the card with the index j is different from the one with the index j+1. We return to the right end of the vector and we reduce k=n until

the card with the index k is bigger than the one with the index j, and then we switch them with each other. Next, within the subsequence containing the cards positioned from j+1 to n, we switch the cards that are situated at an equal distance from the extremes. We repeat the procedure until we find a card positioned k that satisfies the following condition: the card with the index k is bigger than the one with the index j.
For example:
If the elements in the vector are: 1 1 1 1 2 2 2 2 3 3 3 3, then after running the algorithm once, we have: 1 1 1 1 2 2 2 3 2 3 3 3, and after a few more runs we have: 1 2 3 2 1 3 2 1 1 3 3 2.

```
#include <fstream>

using namespace std;
ifstream fin("carti.in");
ofstream fout("carti.out");
int n, v[25], i, ok, j, aux, h, k;
int main()
{
    fin>>n;
    for (i=1; i<=n; i++)
    {
        fin>>v[i];
        fout<<v[i]<<" ";
    }
    do
    {
        j=n-1;
        while (j>0 && v[j]>=v[j+1])
        {
            j--;
        }
        if (j==0)
        {
            ok=1;
            break;
        }
        h=n;
```

Figure 1

```
        while (v[j]>=v[h])
        {
            h--;
        }
        aux=v[j];
        v[j]=v[h];
        v[h]=aux;
        k=j+1;
        h=n;
        while (k<h)
        {
            aux=v[k];
            v[k]=v[h];
            v[h]=aux;
            k++;
            h--;
        }
        fout<<'\n';
        for (i=1; i<=n; i++)
        {
            fout<<v[i]<<" ";
        }
    }
    while (ok==0);
    return 0;
}
```

Figure 2

2. Testing all the combinations

For each possible combination of the 24 cards, we considered every consecutive group of 6 cards and tested if, starting from each of them and moving forward with the number written on the cards, we arrive at the same card, every time.
At first, we started with smaller cases, hoping that we would find a rule so as to solve the actual problem a lot more easily. So we tested the algorithm for 8 cards: 1, 1, 1, 1, 2, 2, 2, 2, then for 12 cards: 1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, and then for 16, and calculated their probability.

1. *For 8 cards*: the probability was 100%. That means that any way you choose the cards from the first group of 2, you arrive at the same card.
2. *For 12 cards*: the probability was 96%, a smaller chance of success.
3. *For 16 cards*: the probability was 81%. Here is where most things changed. The total number of possibilities of combining the cards grew and the time for running the program increased significantly.

When with only 16 cards, the work time increased a lot, so we started searching for a better idea than testing and simulating every single combination possible.
Using the program above, we tested that the number of favourable cases starting with the first card being 1 was the same as the first card being 2, or 3 etc. So we shortened the algorithm, leaving the “do … while” structure when the first card changed. Therefore, the probability for the combinations with the first card being 1 is equal to the final probability.

We divided the process in more parts: using a “for” structure in order to test 10^7 cases at a time and counting how many of them were favourable cases.
It was very important to note the last combination of every round, so that the next round would begin with it. For 10^7 cases the execution time was one minute. A small calculation helped us discover that it took at least 96 hours for the case with 24 cards to be solved. The algorithm we used for simulating each 10^7 cases is shown in ***Figure 3, Figure 4, Figure 5 and Figure 6.***

```
#include <fstream>

using namespace std;
ifstream fin("carti.in");
ofstream fout("carti.out");
unsigned long long n, v[50], i, sep, ok, j,
         aux, h, k, nr[6],a,b,pozitie,x,pozitie1,
         contor,total;
unsigned long long probabilitate,fav;

int main()
{
    fin>>n;
    for (i=1; i<=n; i++)
    {
        fin>>v[i];
    }
    for(i=n+1; i<=2*n; i++)
    {
        v[i]=v[i-n];
    }
    a=1;
    b=(6*n)/24;
    while (a<=n)
    {
        for(pozitie=a; pozitie<a+n;)
        {
            pozitie=pozitie+v[pozitie];
        }
        for (x=a+1; x<=b; x++)
        {
            for(pozitie1=x; pozitie1<a+n;)
            {
```

Figure 3

```
                for(pozitie1=x; pozitie1<a+n;)
                {

                    pozitie1=pozitie1+v[pozitie1];
                }
                if (pozitie1!=pozitie)
                {
                    ok=1;
                    break;
                }
            }
            if (ok==0)
                contor++;
            ok=0;
            a++;
            b++;
        }
        if(contor==n)
            fav++;
        contor=0;
        total=1;
        for (sep=1; sep<=10000000 && v[1]==1; sep++)
        {
            j=n-1;
            while (j>0 &&v[j]>=v[j+1])
            {
                j--;
            }
            if (j==0)
            {
                ok=1;
                break;
            }
```

Figure 4

```
            h=n;
            while (v[j]>=v[h])
            {
                h--;
            }
            aux=v[j];
            v[j]=v[h];
            v[h]=aux;
            k=j+1;
            h=n;
            while (k<h)
            {
                aux=v[k];
                v[k]=v[h];
                v[h]=aux;
                k++;
                h--;
            }
            for(i=n+1; i<=2*n; i++)
            {
                v[i]=v[i-n];
            }
            a=1;
            b=(6*n)/24;
            while (a<=n)
            {
                for(pozitie=a; pozitie<a+n;)
                {
                    pozitie=pozitie+v[pozitie];
                }
                for (x=a+1; x<=b; x++)
                {
                    for(pozitie1=x; pozitie1<a+n;)
                    {
```

Figure 5

```
                    for(pozitie1=x; pozitie1<a+n;)
                    {
                        pozitie1=pozitie1+v[pozitie1];
                    }
                    if (pozitie1!=pozitie)
                    {
                        ok=1;
                        break;
                    }

                }
                if (ok==0)
                    contor++;

                ok=0;
                a++;
                b++;
            }
            if(contor==n)
                fav++;
            contor=0;
            total++;
        }
        fout<<fav<<" "<<total<<'\n';
        for (i=1; i<=n; i++)
        {
            fout<<v[i]<<" ";
        }
        return 0;
}
```

Figure 6

Although we had shortened it twice, it still took too much time to be done. Facing a lack of ideas, we finally thought of something that could help: *What if we actually generated them randomly, so that only the program would know what combinations it chose. Then we could consider the probability for some random cases as being close to the actual probability.*
We generated the combinations using the “random” generator in C++ (algorithm shown in ***Figure 7*** and ***Figure 8***), then tested 10^7 cases, 15 times and made an average. The final result is estimated to be 0.79.

```
#include <iostream>
#include <stdlib.h>
#include <time.h>
using namespace std;
int i,a,f[5],b,v[51],j,pozitie,x,pozitie1,x1,ok,contor,total;
float fav;
int main()
{
    total=0;
    srand (time(NULL));
    for(j=1; j<=35; j++)
    {
        f[1]=f[2]=f[3]=f[4]=f[5]=f[6]=0;
        i=1;
        while(i<=24)
        {
            x1=1+rand()%6;
            f[x1]++;

            if(f[x1]<=4)
            {
                v[i]=x1;
                i++;
            }
        }
        for(i=1; i<=24; i++)
            cout<<v[i]<<' ';
        cout<<endl;

        for(i=25; i<=50; i++)
        {
            v[i]=v[i-24];
        }
```

Figure 7

```
        a=1;
        b=6;
        while (a<=24)
        {
            for(pozitie=a; pozitie<a+24;)
            {
                pozitie=pozitie+v[pozitie];
            }
            for (x=a+1; x<=b; x++)
            {
                for(pozitie1=x; pozitie1<a+24;)
                {
                    pozitie1=pozitie1+v[pozitie1];
                }
                if (pozitie1!=pozitie)
                {
                    ok=1;
                    break;
                }
            }
            if (ok==0)
                contor++;

            ok=0;
            a++;
        }
        if(contor==24)
            fav++;
        contor=0;
        total++;
    }
    cout<<fav/total;
    return 0;
}
```

Figure 8

Conclusion

After testing out a few methods, we reached the conclusion: the probability of arriving at the same card in the group when having 24 cards combined could approximately be 0.79.
We intend to continue testing the algorithm until exhausting all the possible cases.

Match the Maths

2016- 2017

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Teacher: Ph.D. Ioana Cătălina Anton, "Costache Negruzzi" National College of Iași, Romania
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Abstract
Our research deals with the mathematical construction of a soccer ball, that is made of regular pentagons and hexagons. Given the diameter of the ball, we shall approximate the length of the stitch and the area of the ball. Conversely, if the length of the pentagons is given, we shall approximate the size of the ball.

Figure 1. The official FIFA World Cup ball used in Mexico, 1970

The problem

a) Determine the number of pentagons and the number of hexagons on the soccer ball.
b) Find the approximate total length of the stitch?
c) Calculate the sum of the areas of all patches that build it. Is it equal to the area of a sphere of diameter 25 cm? Why?
d) Suppose that a soccer ball and a sphere have the same surface area. Decide which of these two will enclose the highest volume. Why?
e) If the pentagon side on a soccer ball is 4.5 cm, determine the diameter of the ball (by approximation).

For the first point, we saw that we have 12 Pentagons and 20 Hexagons, working with the principle of the symmetry. Secondly, we approximate the soccer ball by a regular truncated icosahedron and after some mathematical calculations, we obtain that the length of the stitch is about 476.1 cm.
In the solution of problem, we have approximated the soccer ball by a truncated regular icosahedron. So, we had to calculate and compare the two areas of the solids and we have founded that $A_{sphere} < A_{ball}$, because the surface of the ball is covered with patches, each patch having edges and so the surface area is not optimized as in the case of a sphere. Doing the same thing with the volume, we have founded that $V_{ball} < V_{sphere}$, where we have approximated the volume of the ball by the volume of a truncated polyhedron.
Finally, we just had to do opposite way because it was given the length of the stitch and we needed to get the diameter.

Solution of the problem

a) After several unsuccessful attempts, we decided to count the edges by the number of vertices as shown down:

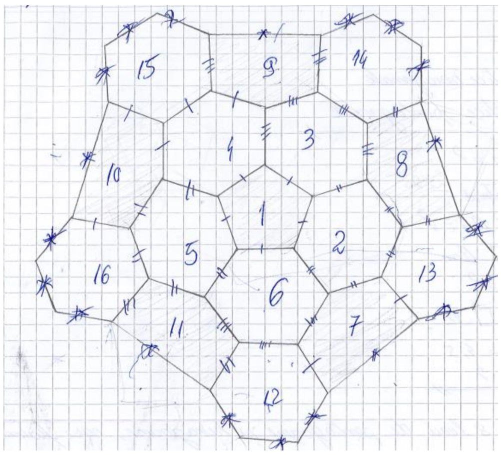

Figure 2. The ball's development in plan

We denote by

$P =$ number of pentagons, $H =$ number of hexagons and $E =$ the number of edges.

We observe that

$$2E = 3V \Rightarrow V = \frac{2}{3} E \quad (1)$$

When counting the edges by the number of faces, we have

$$2E = 5P + 6H \quad (2)$$

By employing the relation (1), we obtain:

$$\frac{2}{3} \times E - E + H = 2 \Leftrightarrow 3H - E = 6 \Leftrightarrow 6H - 2E = 12$$

And thus,

$$6(P + H) - (5P + 6H) = 12 \Leftrightarrow 6P + 6H - 5P - 6H = 12,$$

from where

$$\boxed{P = 12 \textit{ (total number of pentagons)}}$$

A soccer ball is a truncated icosahedron, having 60 vertices. As V = 60, we find from (2) that E = 90 and, from (3),

$$\boxed{H = 20.}$$

Otherwise, as a soccer ball has 32 patches, we have:

$$\boxed{H = 32 - P = 32 - 12 = 20}$$

(b) We shall approximate the soccer ball by a regular truncated icosahedron, as in the figure. Each of the 12 vertices of the regular icosahedron is removed by cutting it with a plane, such that the section (the intersection of the plane with the icosahedron) is a pentagon. We repeat this procedure for all 12 vertices of the icosahedron, such that all the pentagons that we obtain are congruent. The red solids that are cut off from the icosahedron are all pentagonal pyramids.

Each face of the icosahedron is a equilateral triangle. After the cut, the equilateral triangle turns into a regular hexagon. If we denote by l the sidelength of the triangle (which is the side of the icosahedron), then the side of the hexagon is $\frac{l}{3}$ (see Figure (5a) below).

As a consequence, the pentagonal pyramid is regular (see figure (5b) below). Therefore, the volume of the soccer ball can be approximated by the volume of the truncated icosahedron, which is the volume of the icosahedron minus 12 times the volume of a regular pentagonal pyramid (the red cut from the icosahedron).

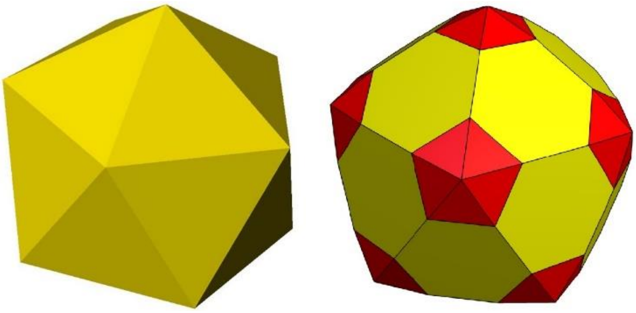

Figure 3. Cutting of the vertices of a regular icosahedron

In order to approximate the total length of the stitch, we shall express the volume of the soccer ball in two ways (see Figure 4), as follows:

- the volume of a sphere with diameter d (V_{sphere}) and
- the volume of a truncated icosahedron $(V_{tr} = V_{ico} - 12V_{pyramid})$.

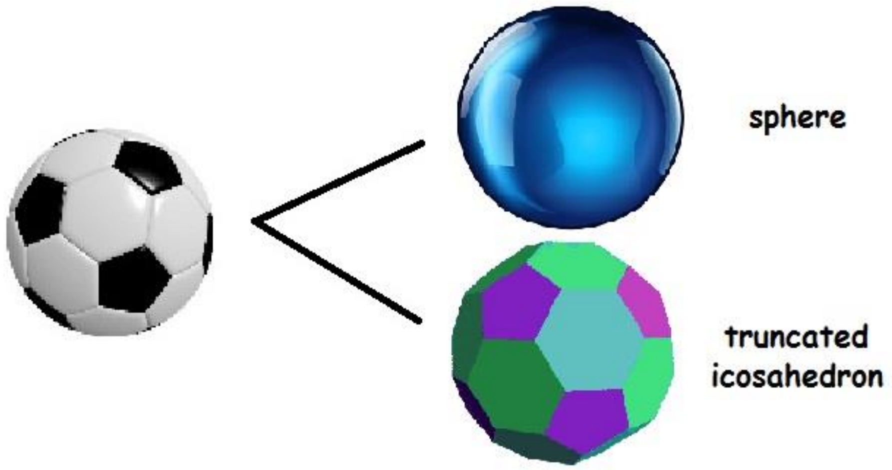

Figure 4. Approximations of a real soccer ball

Firstly, the volume of the sphere of diameter d is:

$$V_{sphere} = \frac{4\pi R^3}{3} = \frac{\pi d^3}{6}.$$

The volume of a regular icosahedron of edge length l is:

$$V_{ico} = \frac{5}{12}(3+\sqrt{5})l^3.$$

The volume of the regular pentagonal pyramid of edge length a is

$$V_{pyramid} = \frac{5+\sqrt{5}}{24}a^3.$$

As here $a = \frac{l}{3}$, we get that

$$V_{pyramid} = \frac{5+\sqrt{5}}{648}l^3.$$

The volume of the truncated icosahedron is

$$\begin{aligned} V_{tr} &= V_{ico} - 12\,V_{pyramid} \\ &= \frac{5}{12}(3+\sqrt{5})l^3 - \frac{5+\sqrt{5}}{54}l^3 \\ &= \frac{125+43\sqrt{5}}{108}l^3. \end{aligned}$$

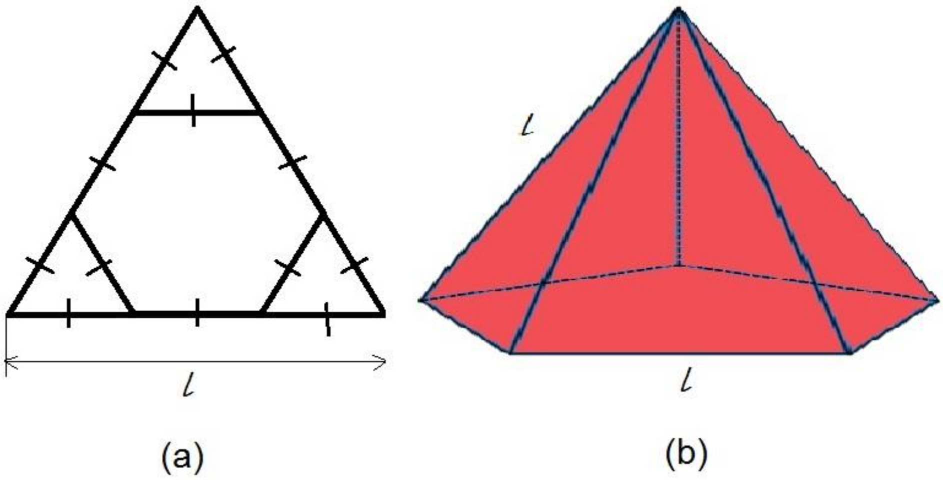

Figure 5. The equilateral triangle and the pentagon pyramid

By equating V_{sphere} and V_{tr} , we obtain:

$$\frac{125+43\sqrt{5}}{108}l^3 = \frac{\pi d^3}{6},$$

hence the side of a triangle is

$$l = d\left(\frac{18\pi}{125+43\sqrt{5}}\right)^{\frac{1}{3}} \cong 15.87cm.$$

The sidelength a of the truncated icosahedron is the length of a side of a pentagon, which is

$$a = \frac{l}{3} \cong 5.29\ cm.$$

As the soccer ball has 90 edges, the total length of the stitch is

$$L = 90\ a \ \cong 476.1\ cm.$$

(b) As calculated previously, the approximate side length a of a pentagon (or hexagon) on the soccer ball for which the volume of the soccer ball (which we approximated with a truncated icosahedron) is equal to the volume of a sphere of diameter 25 cm is $a \cong 5.29\ cm$. We are now comparing the areas of these two solids, the shere and the soccer ball.

Area of a sphere of diameter d is

$$A_{sphere} = \pi d^2 \cong 1963.5\ cm^2$$

On the other side, the area of the soccer ball is

$$A_{ball} = 12 \times \text{Area of a pentagon} + 20 \times \text{Area of a hexagon}.$$

The area of a pentagon of side a is

$$A_p = \frac{a^2}{4}\sqrt{25+10\sqrt{5}}.$$

The area of a hexagon of side a is

$$A_h = \frac{3a^2}{2}\sqrt{3}.$$

Figure 6. A flat soccer ball

Therefore, the area of a soccer ball is

$$A_{ball} = 12A_p + 20A_h = 3a^2(\sqrt{25+10\sqrt{5}} + 10\sqrt{3}) \cong 2031.3\ cm^2.$$

As we can see, $A_{ball} > A_{sphere}$, because the surface of the soccer ball is covered with patches, each patch having edges, and so the surface area is not optimized as in the case of a sphere. Also, a ball which has a smaller area is semificatively faster than a bigger one because of the contact area.

With other words, our calculation is consistent with the following result:

Out of all surfaces that enclose a given volume, the sphere has the smallest surface area.

Figure 7. Kicking a soccer ball

(c) We shall compare now the volumes. We know that fact that the areas of the two solids are equal, so

$$A_{ball} = 3a^2(\sqrt{25+10\sqrt{5}} + 10\sqrt{3}) = \pi d^2 = A_{sphere}.$$

From this, we find that

$$\frac{a}{d} = 1 \Big/ \sqrt{\frac{3}{\pi}\left(\sqrt{25+10\sqrt{5}} + 10\sqrt{3}\right)} \cong 0.2046.$$

On the other hand,

$$V_{sphere} = \frac{4\pi d^2}{3} = \frac{\pi d^3}{6}$$

$$V_{ball} = \frac{125 + 43\sqrt{5}}{4} a^3$$

From the last two relations and using the approximate value of ${}^{a}/_{d}$, we obtain

$$\frac{V_{ball}}{V_{sphere}} = \frac{3}{2\pi}\left(125 + 43\sqrt{5}\right)\left(\frac{a}{d}\right)^3 \cong 0.8297.$$

We observe that this fraction is smaller than 1, and so $V_{ball} < V_{sphere}$.

As we can see, $V_{ball} < V_{sphere}$, due to the fact that the surface of the soccer ball is covered with patches, each patch having edges, and the space in which the air is placed is smaller than in a perfect sphere.

With other words, our calculation is consistent with the following result:

Out of all solids that have the same surface area, the sphere has the largest volume.

(d) We are now given the side length $a = {}^{l}/_{3} = 4.5\ cm$ of a pentagon and we need to find the diameter of the ball. We equate the volumes of the sphere and of the truncated icosahedron:

$$\frac{(125+43\sqrt{5})}{108} l^3 = \frac{\pi d^3}{6} \Rightarrow d^3 = \frac{l^3}{18\pi} \times \left(125 + 43\sqrt{5}\right)$$

$$\Rightarrow d^3 \cong 9816.6$$

and so, $d \cong 21.41 cm$.

Conclusion

In the beginning, we thought that it was easy to build a soccer ball using some mathematical and logical forms. Once we have started to solve the problem and after we have searched for some information about it, we realised that it was much harder. It required a lot of work and exact calculations, a lot of time spend on research and, obviously, quite a few failures. After finishing the solution, we have seen that the problem was just a normal geometry problem, working with a solid having 20 faces (icosahedron), cutting all its vertices and, finally, obtaining the desired soccer ball.

References

- **Bogdan Enescu** – *How to make a soccer ball*, online paper
- **Wikipedia** – *Icosahedron, Platonic solids, isoperimetric property*

This article is written by students. It may include omissions and imperfections.

Moving the Walls

2016 - 2017

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The research task

There is a "wall", composed of a finite sequence of "columns", each containing a finite number of "bricks". We are considering a process of transformation, which consists in separating the first column of the wall from the rest of it, followed by the redistribution of its bricks thus: 1 brick on the second column, 2 bricks on the third column, 3 bricks on the fourth column etc., in the limit of bricks we dispose of. If all the existent columns are "covered" and there are still unused bricks on the 1st column, the process of creating new ones continues.
What can be said about this process of evolution and the different states of it, depending on the initial case?

The results of the research

The initial arrangement (the one given in the problem) repeats from step 2, having a period of length 2 composed of the arrays $[9, 7, 6, 2]$ and $[8, 8, 5, 3]$. The initial arrangement will not be encountered again, given that it is not included in the period. These observations also extend to a general case (any arrangement included in the period will repeat at some point, the arrangements not included in the period will not be encountered again). A particular type of arrays is that of arrangements which repeat themselves, their general formula is $[n, n-1, n-1-2, \dots]$. There is exactly one array which has on its first column n bricks (n is a nonzero natural number) and has the period 1.

The article

We will name an array or arrangement any finite sequence of nonzero natural numbers which shows how many bricks contains each column. An evolution is the creation of a new array, following the given rule. The conclusions from the initial array can be drawn immediately: one can notice that the 4th evolution is the same as the 2nd, therefore all the future evolutions will lead to arrangements equivalent to the ones from the 2nd and the 3rd step. Because none of these is identical to the initial array, it will not be encountered again by following the same rule. We can agree that the period has the length 2 because it contains 2 arrays.

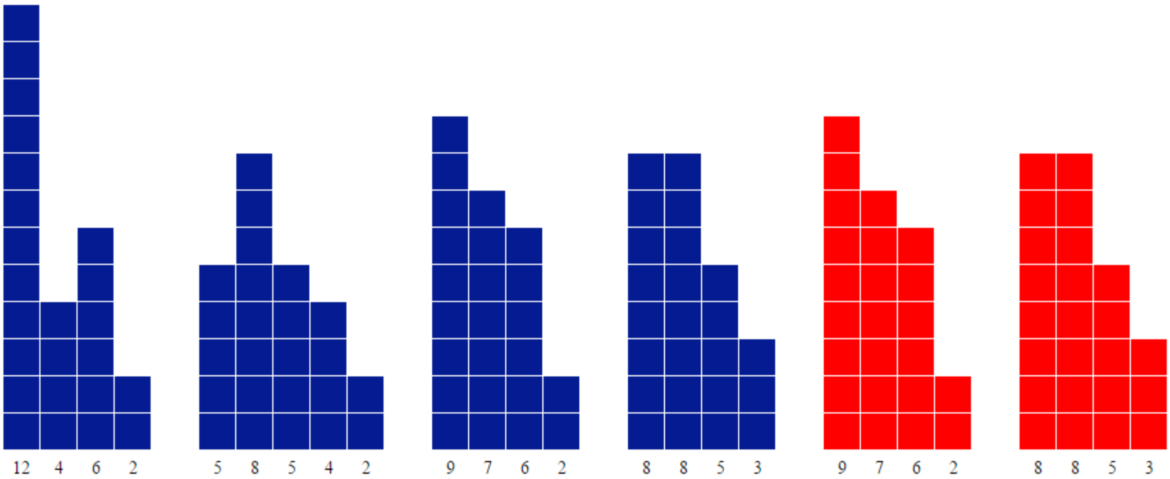

Fig. 1 – The first 5 evolutions of the array $[12, 4, 6, 2]$

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We continued by studying a series of particular arrays – those whose representation is likewise a square matrix or that of an arithmetic progression with the common difference of 1 or -1 – but these don't seem to follow a particular forming rule. What is more, some of them return to the initial arrangement, while the others do not.

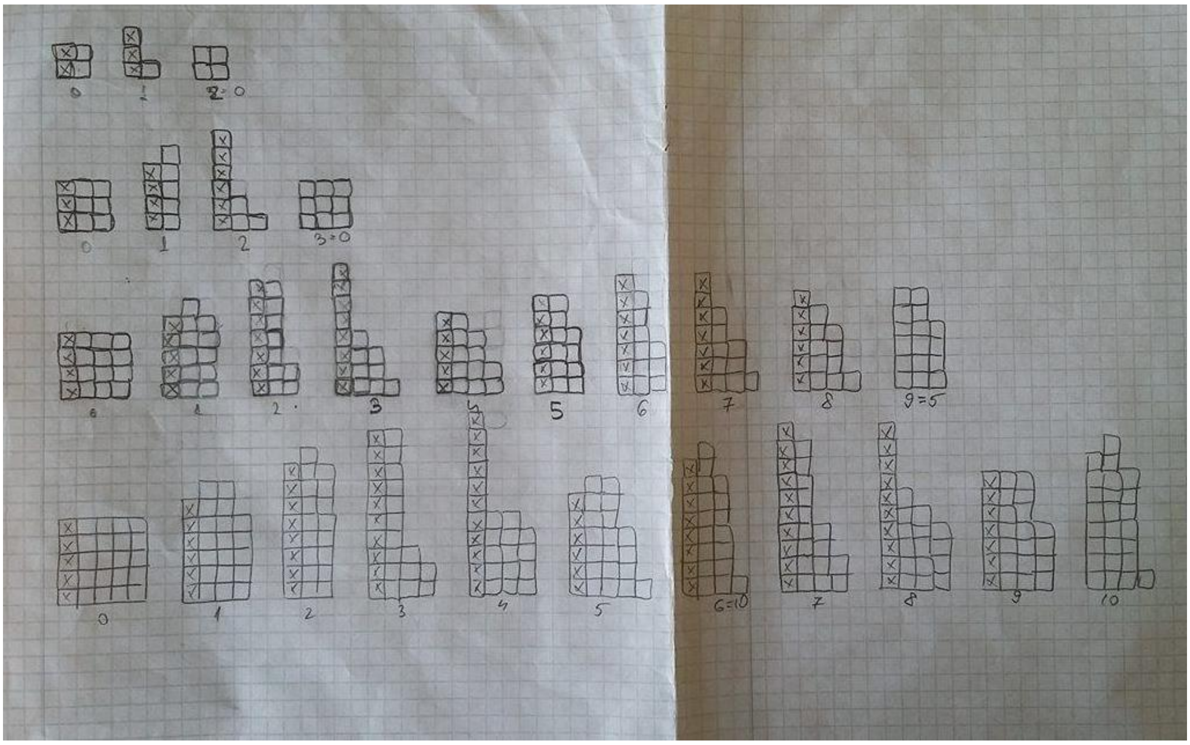

Fig. 2 - The evolutions of square matrix arrangements

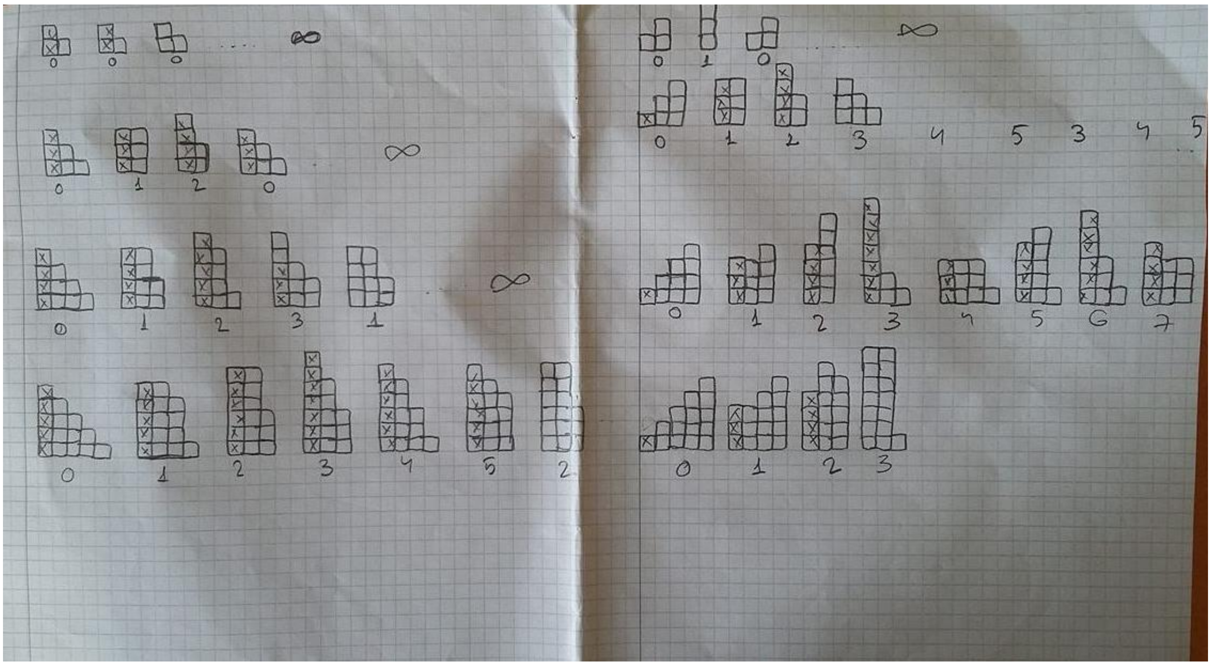

Fig. 3 – The evolutions of progressions with the common difference 1 or -1

Next, we studied the way other arrangements evolve and whether they all have a period. In order to examine a greater number of arrays and evolutions in a short period of time, we wrote a program using JavaScript that displays a visual representation of the evolutions, until a cycle is reached. It is highlighted with red. The program has an array as its input.

We made a second program, so that we could study a greater number of arrays. It creates a table that has on its first column the examined arrangement, on the second one the length of the period, on the third one the step from which the arrays repeat, on the fourth column it shows the first array to repeat and on the last one, the table contains the numbers of order given to every arrangement from the period. The studied arrays start with the length 1, then 2, then 3 and so on, and on each column, there are between 1 and N bricks. So, the total number of generated arrays is $\sum_{k=1}^{n} N^k$. The number of order given is obtained thus:

$\sum_{k=0}^{n-1} c_{n-k} * N^{n-k-1}$, where c_k represents the number of bricks on the column k. After the results obtained by the program, we noticed that a high number of arrangements (a few thousands are studied in less than one minute) reaches a period.

Array	Period	Array repeats from step	First array that repeats	Numbers for arrays in period
1	1	0	1	1
2	2	0	2	2, 21
3	2	0	3	3, 22
4	2	2	3, 1	42, 61
5	1	2	3, 2	62
6	3	2	3, 3	63, 82, 1241
7	3	2	3, 3, 1	83, 1261, 1641
8	3	2	3, 3, 2	84, 1262, 2041
9	3	2	3, 3, 3	85, 1263, 2441
10	3	6	4, 5, 1	1663, 1701, 2461
11	3	6	4, 5, 2	1702, 2063, 2481
12	3	6	4, 5, 3	1703, 2463, 2501
13	3	10	6, 5, 2	2103, 2483, 2502
14	1	10	6, 5, 3	2503
15	4	7	6, 6, 3	2504, 2523, 2903, 50061
16	4	7	6, 6, 4	2524, 2923, 50081, 58061
17	4	4	7, 7, 3	2525, 2943, 50082, 66061
18	4	4	7, 7, 4	2944, 50501, 58082, 66461
19	4	4	7, 7, 5	2945, 50502, 66082, 66861
20	4	4	7, 7, 6	2946, 50503, 67261, 74082

Fig. 4 – The first lines of the generated table

An important observation is that the number of bricks from an arrangement is constant, so there is a finite number of ways in which these can be placed in columns. It is equal to the number of ways in which a nonnegative natural number N can be written as the sum of nonnegative natural numbers, where the order of the addends does matter. In order to find the number of sums, we can imagine that we have to place a plus sign or a comma in every box from $\overbrace{1\square1\square\dots\square1\square1}^{N}$. Because there are $N-1$ boxes and a binary choice for each, there are 2^{N-1} possible arrangements. Given that their number is finite, it is evident that at some point one array will be identical to a previous one, so the period is formed. This proves that in every process of evolution there will exist a repeating cycle.

Last but not least, we studied a series of arrays with the property that they have the period 1 (that is, they repeat themselves). For each nonnegative natural number n, there is exactly one arrangement with n bricks on its first column and with the period 1. Its general formula is $[n, n-1, n-1-2, \dots]$; the process of adding new elements in the array finishes when one element is less than or equal to its position in the array.
At the evolution 0, we denote L the numbers of columns. In the example of $[3,2]$, we have $L=2$. Then, S represents the number of solutions for L fixed. For example, for $L=2$, we have $S=2$, which are $[2,1]$ and $[3,2]$. Finally, A represents the smallest height of the first column for S solutions. For $L=2$ and $S=2$, A is 2 because between $[2,1]$ and $[3,2]$, the smallest number of bricks on the first column is 2.

We observe that the number of bricks on the first column of the last array of length L (with the period 1) is equal to $1+2+\dots+L=\frac{L(L+1)}{2}$. However, if we wish to find the number of bricks on the first column of the arrangements with L columns, the formula is:
$A_L=\frac{L(L-1)}{2}+1.$

▶ [1]	▶ [15, 14, 12, 9, 5]	▶ [29, 28, 26, 23, 19, 14, 8, 1]
▶ [2, 1]	▶ [16, 15, 13, 10, 6, 1]	▶ [30, 29, 27, 24, 20, 15, 9, 2]
▶ [3, 2]	▶ [17, 16, 14, 11, 7, 2]	▶ [31, 30, 28, 25, 21, 16, 10, 3]
▶ [4, 3, 1]	▶ [18, 17, 15, 12, 8, 3]	▶ [32, 31, 29, 26, 22, 17, 11, 4]
▶ [5, 4, 2]	▶ [19, 18, 16, 13, 9, 4]	▶ [33, 32, 30, 27, 23, 18, 12, 5]
▶ [6, 5, 3]	▶ [20, 19, 17, 14, 10, 5]	▶ [34, 33, 31, 28, 24, 19, 13, 6]
▶ [7, 6, 4, 1]	▶ [21, 20, 18, 15, 11, 6]	▶ [35, 34, 32, 29, 25, 20, 14, 7]
▶ [8, 7, 5, 2]	▶ [22, 21, 19, 16, 12, 7, 1]	▶ [36, 35, 33, 30, 26, 21, 15, 8]
▶ [9, 8, 6, 3]	▶ [23, 22, 20, 17, 13, 8, 2]	▶ [37, 36, 34, 31, 27, 22, 16, 9, 1]
▶ [10, 9, 7, 4]	▶ [24, 23, 21, 18, 14, 9, 3]	▶ [38, 37, 35, 32, 28, 23, 17, 10, 2]
▶ [11, 10, 8, 5, 1]	▶ [25, 24, 22, 19, 15, 10, 4]	▶ [39, 38, 36, 33, 29, 24, 18, 11, 3]
▶ [12, 11, 9, 6, 2]	▶ [26, 25, 23, 20, 16, 11, 5]	▶ [40, 39, 37, 34, 30, 25, 19, 12, 4]
▶ [13, 12, 10, 7, 3]	▶ [27, 26, 24, 21, 17, 12, 6]	▶ [41, 40, 38, 35, 31, 26, 20, 13, 5]
▶ [14, 13, 11, 8, 4]	▶ [28, 27, 25, 22, 18, 13, 7]	▶ [42, 41, 39, 36, 32, 27, 21, 14, 6]

Fig. 5 – The first 42 arrangements that repeat themselves

L	S	A
1	1	1
2	2	2
3	3	4
4	4	7
5	5	11
6	6	16
7	7	22

Table 1 – First examples of arrangements, with the number of columns (L), the number of solutions (S) and the bricks on the first column of the first solution (A)

This article is written by students. It may include omissions and imperfections.

Multiple Reflections in Plane Mirrors

2016- 2017

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Our research topic
Finding out the number of images formed between two intersected mirrors.

Brief presentation of the conjectures and results obtained

This problem is solved in the textbooks we see only for the case in which the angle formed by the two mirrors is equal to $180^{o}/n$, where n is an integer number. In this case the number of images created does not depend on the position of the object that is between the mirrors. Our team decided to study this problem in the general case. We found out that if the angle has a different value than the one mentioned previously, the number of images depends not only on the angle between the mirrors, but also on the position of the object. Using GeoGebra, we have managed to find a way to calculate the number of images formed in each case.

1. Representing the multiple reflexions using polar coordinates

To make a model of the multiple reflexions created between two intersecting mirrors we had represented the source object using a point named *S* (r cos t, r sin t). We represented the mirrors using two segments that connected the origin of the coordinate system with the two points *A*(d cos u/2, d sin u/2) and *B*(d cos (-u/2), d sin (-u/2)). Since the object was supposed to be placed between the mirrors, it was necessary for the absolute value of the angle *t* to be smaller than u/2. But, basically, the system of mirrors was built introducing the next instructions in the input bar of the *GeoGebra* app:

- O = (0,0); u = 40^{o}; d = 10;
- A = (d*cos (u/2), d*sin (u/2)); B = (d*cos (-u/2), d*sin (-u/2));
- t = 10^{o}; r = 6; S = (r*cos (t), r*sin (t));

To change the value of the angle *u* between the two mirrors we will use a slider, which we can show by selecting the circle next to the variable *u* in the algebra panel of the app (Fig. 2). Same goes for the representation of the angle *t*, which we will later use for moving the *S* point between the mirrors. To keep point *S* between the two mirrors we set the extreme values of *t* to –*u*/2 and *u*/2 through the *Preferences* panel of the app (Fig. 3)

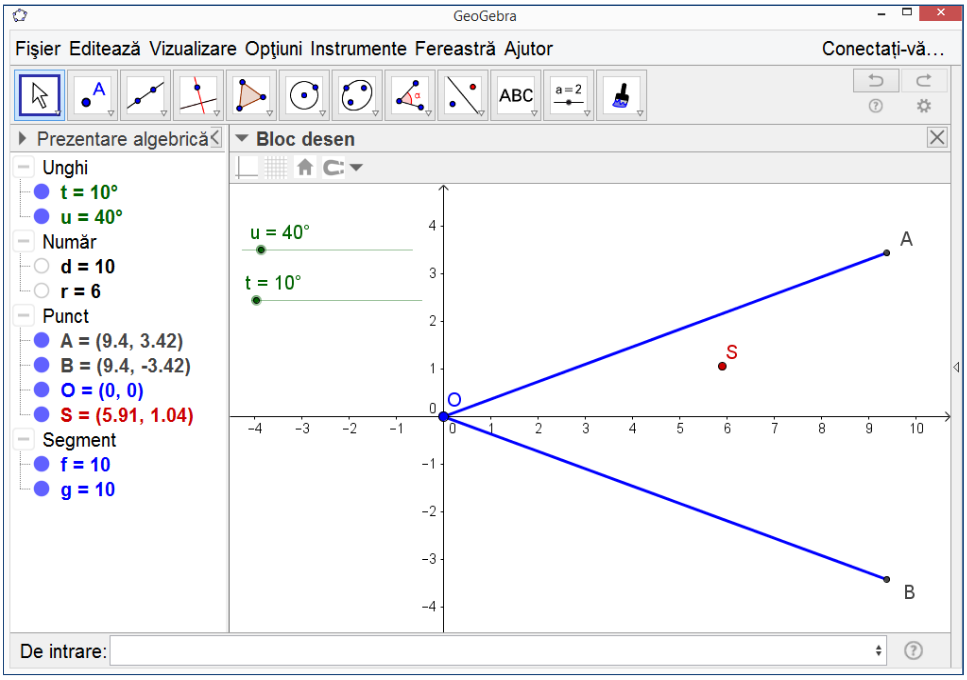

Fig. 1. Representing the mirror system using *GeoGebra*

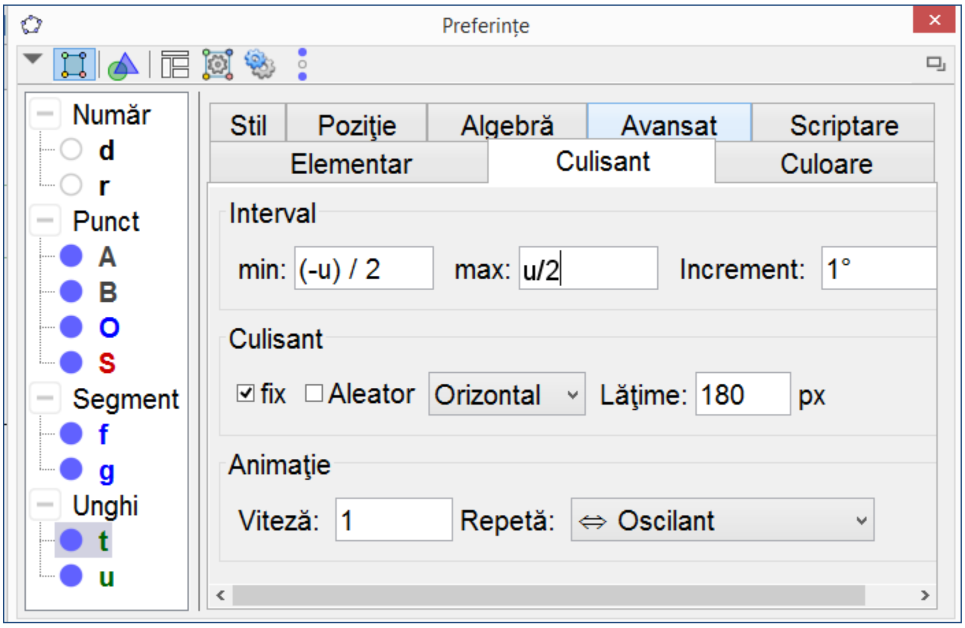

Fig. 2. Setting the extreme values of the variable *t*

If point S_1 is the image of S in the OA mirror, using the congruence of the triangles OMS and OMS_1 we have demonstrated that $OS_1=OS$ and that the size of the angle xOS_1 is equal to u-t. If S_2 is the image of S_1 in mirror OB, S_3 is the image of S_2 in OA etc., an inductive reasoning leads us to the conclusion that the size of angle xOS_i is $i*u$-t. Since the angles xOS_i, are orientated counter clockwise if i is an odd number, and clockwise if i is even, it results that the angular coordinate of image S_i is (-1)^(i+1)*($i*u$-t), while the radial coordinate, OS_i, is equal to $OS = r$. This observation allows us to represent all S_i images using a *Sequence* command.

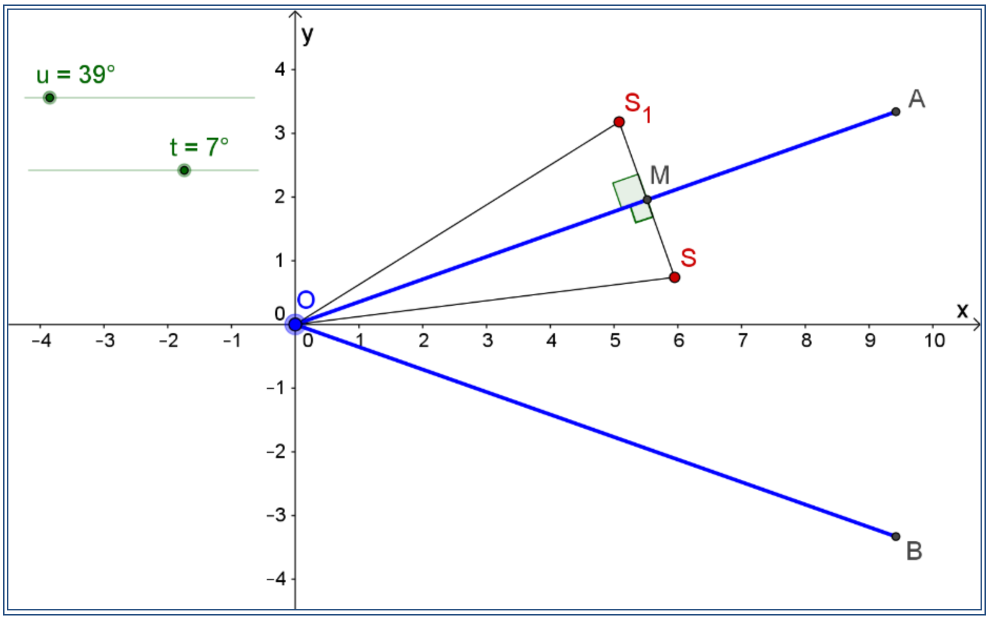

Fig. 3. Finding out the coordinates of the first reflexion of object *S* in the *OA* mirror

The structure of the *Sequence* command is automatically shown when introducing the first characters in the input bar (Fig.5). All we have to do is to complete the four positions that are indicated. We will complete the <*Expression*> with the polar coordinates deduced previously, (r*cos((-1) ^ (i+1)*(*i*u-t*)),r*sin((-1) ^ (i+1)*(*i*u-t*))), <*Variable*> with *i*, <*Start Value*> with 1 and <*End Value*> with *n*. After entering the command there will appear a window that asks us if we want to create a slider for the variable *n*. With the help of this slider we can control the number of S_i images that are displayed. (Fig. 6)

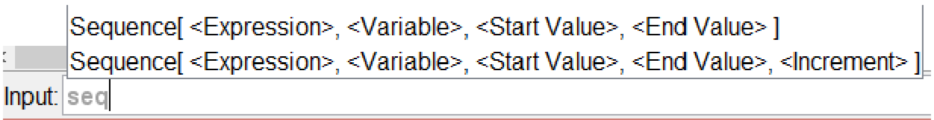

Fig. 4. The structure of the *Sequence* command

Using a similar reasoning that starts with discovering the coordinates of image T_1 of the object *S* in the *OB* mirror allows us to inductively find out the coordinates of all T_i images and represent them through a single command:
*Sequence [(r *cos((-1)^i *(i *u + t)), r* sin((-1)^i *(i* u + t))), i, 1, n]*.

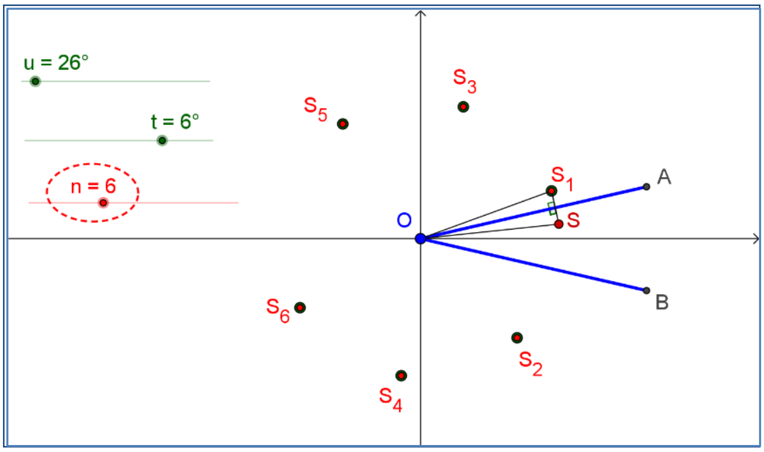

Fig. 5. Showing *n* successive images in the model built with *GeoGebra*

2. Finding out the number of images between two intersecting mirrors

The model built allows us to study some interesting situations. If the angle between the two mirrors is u=180º/k, there will appear k-1 different reflexions of the object, S_1,S_2, ...S_{k-1} while point S_k overlaps with object S, point S_{k+1} overlaps with S_1, S_{k+2} overlaps with S_2, etc.

If the angle isn't u=180º/k, with k integer, for establishing the number of images created it is necessary to know that a new S_i reflection can form only if the previously reflected image S_{i-1} was located in front of the mirror in which it should reflect. Considering the angular coordinates we found out previously, this conditions returns to

$$iu - t < 180^o - \frac{u}{2} \qquad (1)$$

and from here we establish that the number of S_i images created is

$$n' = \left[\frac{180^o}{u} + \frac{t}{u} + \frac{1}{2} \right], \qquad (2)$$

where [x] is the integer part of x. A similar reasoning leads us to the conclusion that the number of T_i images that form is

$$n'' = \left[\frac{180^o}{u} - \frac{t}{u} + \frac{1}{2} \right] \qquad (3)$$

and in the end there will be $N = n' + n''$ images. Relations (2) and (3) help us with finishing the *GeoGebra* simulation after entering the next two commands:

- n' = floor (180º/u+t/u+1/2)
- n'' = floor (180º/u-t/u+1/2)

followed by replacing the value of n with n' and respectively n'' in the two *Sequence* commands.
Right now, the model works for every value of the angle t and u and allows us to investigate ***all*** the possible situations.

3. Does the number of reflexions created depend on the position of the object between the mirrors?

With the help of the simulation built we have observed that if the angle formed by the two mirrors isn't u=180º/k, with k integer, the number of images also depends on the position of the object between the mirrors, more precisely on the angular coordinate of object S. We tried to find out the limit values of the angular coordinate for which the number of images changes. Since $|t|<u/2$, the component t/u+1/2 from the relation (2) is always found in the (0,1) interval. If 180º/ $u = q + f$, where q is the integer part, and f is the fractional part of number 180º/u, then the (2) relation shows us that number n' can be equal to q or $q+1$, which would mean that the value changes if $f+t/u+1/2=1$, therefore if $t = t_1 = u\,(1/2-f)$. A similar reasoning based on relation (3) leads us to the conclusion that the number of T_i images formed modify when the angular coordinate t of object S has the value $t = t_2 = -u\,(1/2-f)$.
To represent the borders established previously we have entered the next *GeoGebra* commands:

- q = floor(180º/u);
- f = 180º/u-q;
- t_1= u (1/2-f);
- t_2= - u (1/2-f);
- C = (d cos(t_1), d sin (t_1));

- D = (d cos(t_2), d sin (t_2));

By moving the *S* point, we can confirm that our reasoning was correct: if we cross the (OC semi-line there will appear one more image (Fig.7).

The algebraic expression of angles t_1 and t_2, formed by the border semi-lines with the *Ox* axis allows us to identify with ease the special cases. If *f*, the fractional part of number 180°/ *u* is equal to ½, the two borders overlap with *Ox*, which is the bisecting line of the angle created by the mirrors. This happens if the angle created by the mirrors is in the form of *u=360°/(2k+1).* In this case *n'* and *n''* are equal to *q*+1 if point *S* is found on the bisector of angle *u*. Otherwise, one of the number *n'* or *n''* is equal to *q*, while the other is *q+1*, *2q*+1 images being visible. Surprisingly, when point *S* is found on the bisector, there are *n' + n''* = *2q*+2 images that aren't visible, while only *2q* images are so. The explanation can be revealed by analysing the angular coordinates of images S_i and T_i. In this case the last two S_i images overlap, just like the last T_i images, which can be justified through the periodicity of sine and cosine functions. An exemplification of this situation is shown in picture 8.

The expression of angles t_1 and t_2 shows us that if *f*, the fractional part of number 180°/ *u*, is smaller than 1/2, $t_1>0$ and $t_2<0$, and if *f* is bigger than ½, the signs reverse. This means that, in the first case, the number of images created is smaller when *S* is located between the two border semi-lines, while in the second case, the number of images is bigger when *S* is between the borders.

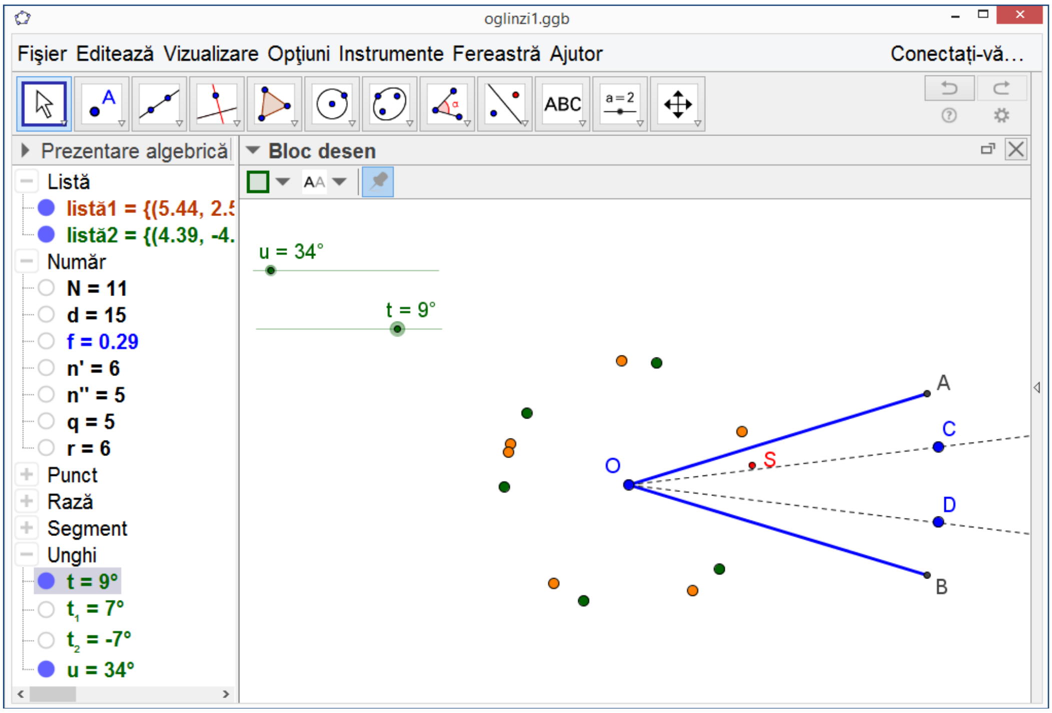

Fig. 6. The number of reflected images modifies when *S* crosses the *OC* border

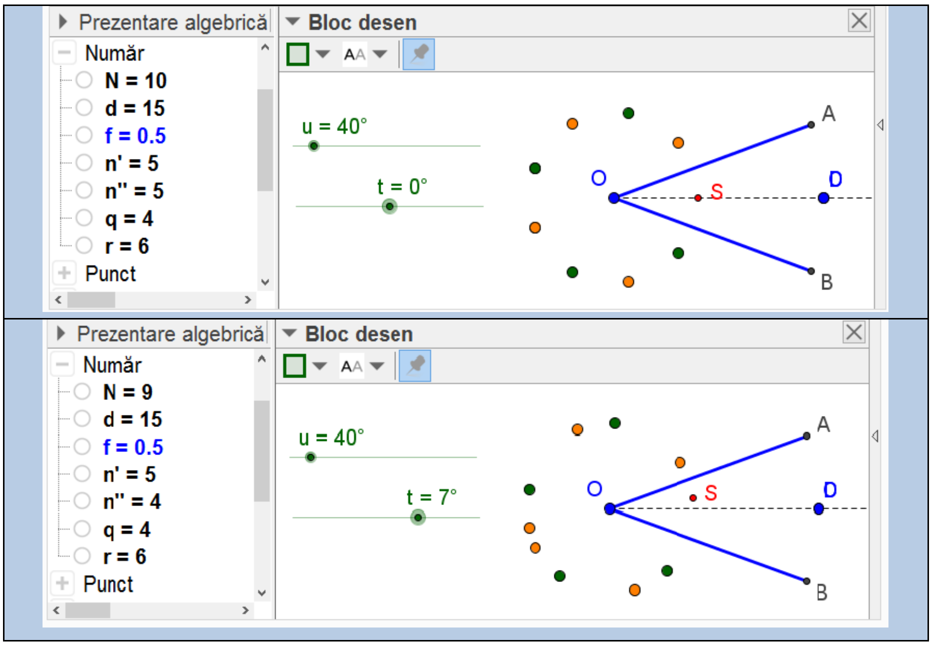

Fig. 7. The case when the angle is in the form of 360°/(2*k*+1)

4. Conclusions

After discussing with some Physics teachers, we realized that the situations simulated by us in the *GeoGebra* are not known, which made them quite surprised. Building the interactive model with *GeoGebra* allowed us to solve a problem in its general case with ease, which would have been rather hard to solve without the app. Physics textbooks take into consideration only the case in which the angle formed by the two mirrors is a fraction in the form of 1/n of a straight angle, even though in a note from 1902, M. G. Lloyd showed that this case was just a particular one, the optic system behaving differently in the general case [1]. The advantages of using *GeoGebra* in solving this problem were:

- easily building a representation of the system in its general case;
- easily modifying the parameters of the system with the help of the sliders that the app creates for every variable;
- the possibility of observing the correspondence between the algebraic form and geometric form of a mathematical object through the algebraic and graphic panel of the app;
- the possibility of checking if the calculations were correct with the help of the graphic representation;
- the possibility of identifying and studying with ease the special configurations of the system.

The following table contains the results we have found.

$f = \left\{\frac{180°}{u}\right\}$	*u* (the angle between mirrors)	The limit angles *t1* și *t2*	The angular coordinate *t* of object *S*	The number of images $q = \left[\frac{180°}{u}\right]$		
				ns	nT	The number of observed images
f=0	u=360º / (2q)	t1 = - u/2 t2 = u/2	t1<t<t2	q	q	2q-1
f between 0 și 1/2	u between 360º / (2q+1) and 360º / (2q)	t1<0<t2	-u/2<t< t1	q+1	q	2q+1
			t1<t<t2	q	q	2q
			t2< t<u/2	q	q+1	2q+1
f=1/2	u=360º / (2q+1)	t1=t2=0	-u/2<t<t1=t2	q+1	q	2q+1
			t=t1=t2=0	q+1	q+1	2q
			t1=t2<t<u/2	q	q+1	2q+1
f between 1/2 și 1	u between 360º / (2q+2) and 360º / (2q+1)	t2<0<t1	-u/2<t<t2	q+1	q	2q+1
			t2< t< t1	q+1	q+1	2q+2
			t1<t<u/2	q	q+1	2q+1

5. References

[1] M. G. Lloyd (1902). Note on the Multiple Images Formed by Two Plane Inclined Mirrors. *Science*, *6*(399), pp 316-317.

This article is written by students. It may include omissions and imperfections.

Paths on a Grid

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Keywords: path, grid, counting, product rule, the inclusion-exclusion principle.

Presentation of the research topic

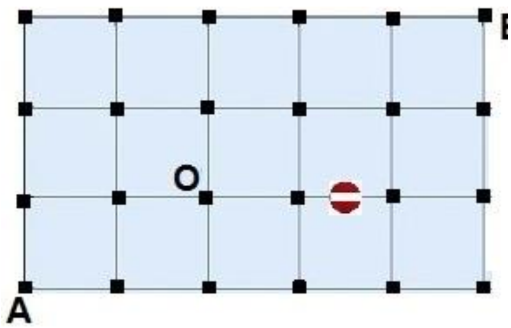
In this article, we are counting paths that join two opposite corners of a rectangular grid. The grid is supposed to represent the street map of a town centre. Two grid sizes and various traffic restrictions are considered.

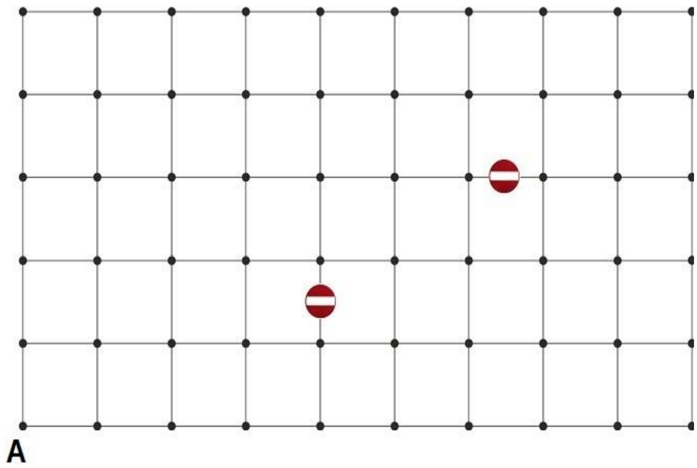
Brief presentation of the conjectures and results obtained

In the following, we will agree that:

node = the intersection point of two lines of the map;
street = a segment that unites two adjacent nodes of the map;
path = a succession of streets.

Diagrams (α) and (β) below represent the mapping of the streets from the centre of two towns, *Little Rock* and, respectively, *Big Hill*. Each side of a square represents a new street, and in each node one touristic attraction exists. On each of these streets, you can walk (as a pedestrian) in any direction or you can drive only in two directions, rightward or upward.





(α): The *Little Rock* town centre (β): The *Big Hill* town centre

We intend to address the following problems:

(a) In how many ways can one drive from point *A* to point *B* of the *Little Rock* town without counting the possible restrictions from the route?

(a_1) Same question as in (a), but with the restriction to pass through attraction *O*.

(a_2) Same question as in (a), taking into account that the traffic is closed on the street marked in Figure (α) (in other words, the respective square becomes inaccessible).

(a_3) For which street, supposing it is closed, the number of ways of driving from *A* to *B* is minimal? Is this the only street having this property?

On two of the *Big Hill* town streets (marked in the figure) access is forbidden for cars and pedestrians alike.

(b) In how many ways can one drive from point *A* to point *B* of *Big Hill*, taking into account the restrictions?

(c) Two walkers, *Alin* and *Bianca*, depart simultaneously from points *A* and *B*, respectively. *Alin* may move only one square right or upwards, while *Bianca* may move only one square left or downwards (when looking at the grid from the same perspective). For each of them, the choice of direction is made with equal probabilities in every intersection. What are the odds that *Alin* and *Bianca* meet?

(d) Apart from the two above mentioned restrictions, other streets in the town will be under repair and will be inaccessible for a while. Find out the maximum number of streets that can be under repair simultaneously, so that all the touristic objectives are connected to point *A* by at least one path. We maintain the original rules of movement. The path starting at *A* may end in any point (touristic objective).

We shall solve the traffic problems in two ways: analytically, by using tools from combinatorics and numerically, using C++ programming.

Theoretical notions

A. Let $n \in N^*$. The product $n! = 1 \cdot 2 \cdot ... \cdot n$ is called *n factorial*. By convention, $0! = 1$. We observe that $n! = (n-1)! \cdot n$, for any $n \in N^*$.

Let $n \in N^*$., $k \in N, k \leq n$. The number of subsets which have k elements of a set with n elements is named *combinations of n elements taken k at a time* and it is written as $\binom{n}{k}$.

Of course, $\binom{n}{k} = \binom{n}{n-k}$, because choosing a subset with k elements is equivalent to choosing a subset with $n-k$ elements to be left out.

It is known that $\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$, for any $n \in N^*$. and any $k \in N, k \leq n$.

B. Theorem (*Principle of inclusion-exclusion for two sets*) If A and B are two finite sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

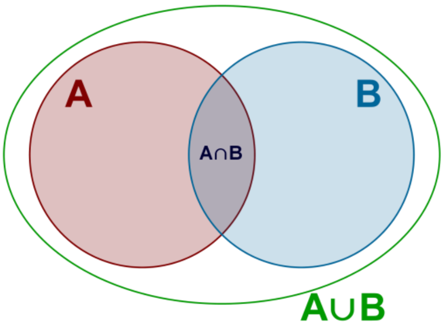


Figure 1

In other words, given 2 finite sets, the number of elements belonging to at least one of the sets is equal to the sum of the number of elements in each set, from which we subtract the number of elements common to the 2 sets.

C. *The multiplication principle (the product rule)*. If A and B are two sets and $A \times B$ notes their Cartesian product (i.e. $A \times B = \{(a,b) \mid a \in A, b \in B\}$), then

$$|A \times B| = |A| \cdot |B|. \tag{0.1}$$

Otherwise stated, given two finite sets, the number of ways in which one can form ordered pairs in which the first element is in the first set and the second element is in the second set is equal to the product of the number of elements of the two sets.

Formula (1.1) generalizes to a finite number of sets $A_1, A_2, \ldots, A_n$:

$$|A_1 \times A_2 \times \ldots \times A_n| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_n|.$$

The solution

(a) **In how many ways can one drive from point *A* to point *B* of the *Little Rock* town without counting the possible restrictions from the route?**

<u>First method</u>

We are computing the number of paths starting from *A* and ending in *B*, satisfying the requirements in the problem. Figure 2 illustrates how we must proceed:

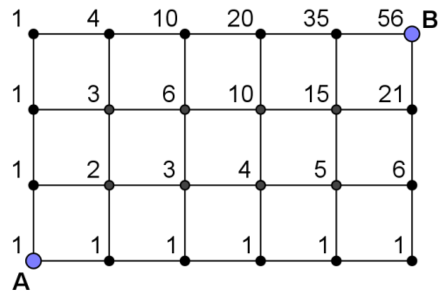


Figure 2

So we observe that the number of paths from *A* to *B* which respect all the conditions is *56*.

<u>Second method</u>

Because a car can move across streets only in two directions, rightward or upward, to arrive from point *A* to point *B* we must go five streets to the right and three streets upward, in a certain order. To describe the path, we can picture that we have five R letters and three U letters. When the car goes to the right we write R and when it goes upward we write U. For each possible route, we obtain a unique sequence of eight letters. Figure 3 demonstrates this.

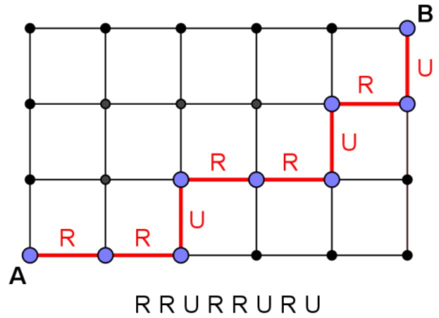


Figure 3

To calculate the number of possible route returns we calculate the number of different sequences which are written with five R letters and three U letters.

We observe that to construct a sequence knowing the position of the three U letters is enough. So, it is enough to know which are the 3 numbers from 1 to 8 which correspond to the position of the U letters in the sequence. These 3 numbers form a subset with 3 elements from the set $P=\{1,2,3,4,5,6,7,8\}$. In conclusion, the number of possible routes is the number of subsets with 3 elements from the set P with 8 elements, that is $\binom{8}{3}=\frac{8!}{3!\cdot 5!}=\frac{6\cdot 7\cdot 8}{1\cdot 2\cdot 3}=56$.

More general, if the map was a rectangle with length composed by a number of streets and the width being a b number of streets, the number of routes from which a driver can choose to get from one corner of the map to the opposite corner would be $\binom{a+b}{a}=\binom{a+b}{b}=\frac{(a+b)!}{a!\cdot b!}$.

<u>Third method: C++ program</u>

Let us consider the grid from Figure 4:

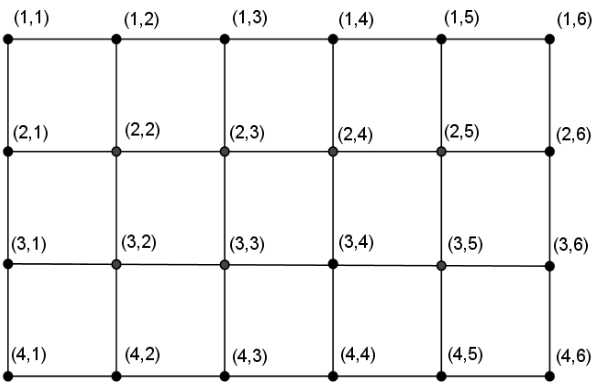


Figure 4

main.cpp

```cpp
#include <iostream>
using namespace std;
int i,j,n,m,h[10][10];
int main()
{
    cin>>n>>m;
    for(i=n; i>=1; i--)
        for(j=1; j<=m; j++)
            if(j==1||i==n)
                h[i][j]=1;
            else
                h[i][j]=h[i+1][j]+h[i][j-1];
    cout<<h[1][m];
    return 0;
}
```

"E:\Info\a1 mathENJeans\bin\...

```
4 6
56
Process returned 0 (0x0)   exec
Press any key to continue.
```

For n=4 and m=6, we obtain 56. In the program n is the number of vertical nodes, m is the number of horizontal nodes and the element $h[i][j]$ of the matrix h stores how many paths arrive in the node (i, j), with $1 \le i \le n$ and $1 \le j \le m$ (node coordinates are given in Figure 4).

(a_1) Same question as in (a), but with the restriction to pass through attraction O.

First method

Each path from A to B is composed by a path from A to O and a path from O to B (paths which respect the conditions) (Figure 5).

The path's map from A to O is a rectangle with length composed by 2 streets and width composed by a street, so the number of possible routes is $\binom{1+2}{1} = \binom{3}{1} = 3$. The path's map from O to B is a rectangle with length composed by 3 streets and width composed by 2 streets, so the number of possible routes is $\binom{2+3}{2} = \binom{5}{2} = \frac{5!}{2! \cdot 3!} = 10$ (see Figure 6).

In conclusion, according to *the product rule*, the number of routes from A to B, which pass through O, is $3 \cdot 10 = 30$.

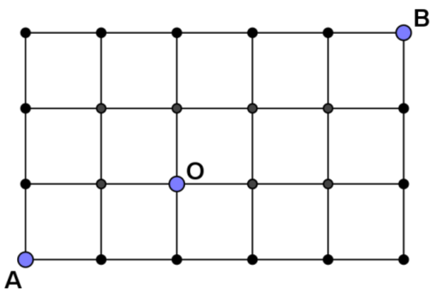


Figure 5

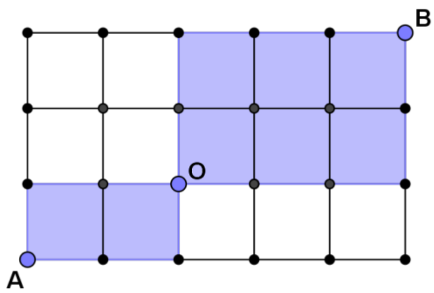


Figure 6

Second method : C++ program

In addition to the first program, (x, y) is the node we must pass through.

```cpp
#include <iostream>
using namespace std;
int i,j,x=3,y=3,n,h[10][10],m;
int main()
{
    cin>>n>>m;
    for(i=n; i>=x; i--)
        for(j=1; j<=y; j++)
            if(j==1||i==n)
                h[i][j]=1;
            else
                h[i][j]=h[i+1][j]+h[i][j-1];
    for(i=x; i>=1; i--)
        for(j=y; j<=m; j++)
            if(i==x&&j>y)
                h[i][j]=1;
            else
                if(i<x&&j==y)
                    h[i][j]=1;
                else
                    if(i<x&&j>y)
                        h[i][j]=h[i+1][j]+h[i][j-1];
    cout<<h[x][y]*h[1][m];
    return 0;
}
```

```
4 6
30
Process returned 0 (0x0)
Press any key to continue.
```

For n=4 and m=6, we obtain *30 paths*.

(a_2) Same question as in (a), taking into account that the traffic is closed on street *CD* (see Figure 7) (in other words, the respective square becomes inaccessible).

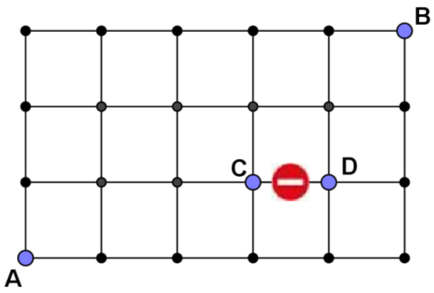


Figure 7

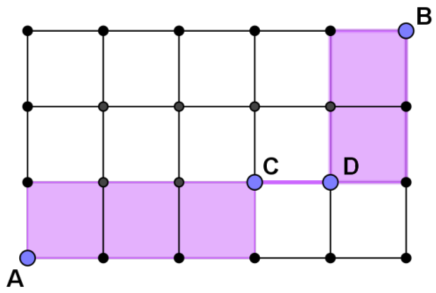


Figure 8

First method

One may count the number of paths that verify the restriction by subtracting the number of paths from *A* to *B* (which is *56*, according to point (a)) and the number of paths that contain street *CD* closed to traffic.

A path that contains street *CD* is composed of a path from *A* to *C*, street *CD* and a path from *D* to *B* (see Figure 8).

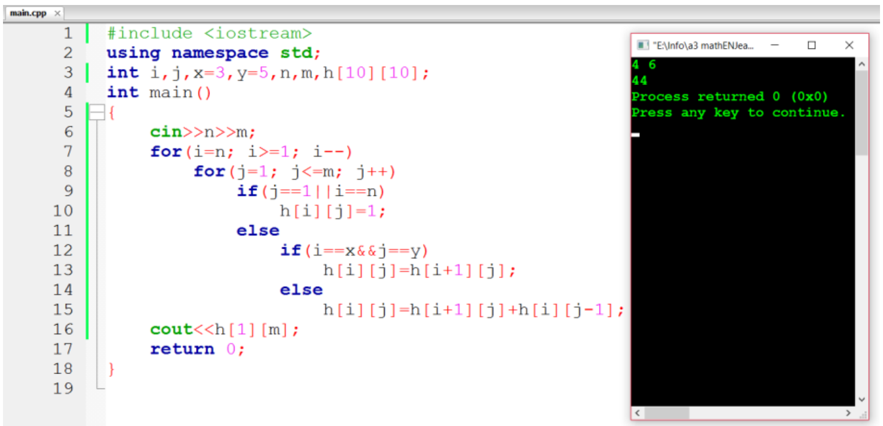
The number of paths from *A* to *C* is $\binom{3+1}{1}=\binom{4}{1}=4$. The number of paths from *D* to *B* is $\binom{2+1}{2}=\binom{3}{2}=3$.

Summing these, there are $4\cdot1\cdot3=12$ paths which contain street *CD*.

In conclusion, the number of paths which avoid street *CD* is $56-12=44$.

Second method : C++ program

In this program, the coordinates of point *C* are (x,y) and the coordinates of point *D* are, of course, $(x,y+1)$.



```
#include <iostream>
using namespace std;
int i,j,x=3,y=5,n,m,h[10][10];
int main()
{
    cin>>n>>m;
    for(i=n; i>=1; i--)
        for(j=1; j<=m; j++)
            if(j==1||i==n)
                h[i][j]=1;
            else
                if(i==x&&j==y)
                    h[i][j]=h[i+1][j];
                else
                    h[i][j]=h[i+1][j]+h[i][j-1];
    cout<<h[1][m];
    return 0;
}
```

For n=4 and m=6, we obtain *44 paths*.

(a_3) For which street, supposing it is closed, the number of ways of driving from *A* to *B* is minimal? Is this the only street having this property?

<u>First method : C++ program</u>

```
#include <iostream>
using namespace std;
int i,j,x,y,nrmin=57,xmin1,xmin2,ymin1,ymin2,nr=0,n,m,h[10][10];
int main()
{
    cin>>n>>m;
    for(x=1; x<=n; x++)
        for(y=1; y<=m; y++)
        {for(i=n; i>=1; i--)
                for(j=1; j<=m; j++)
                    if(j==1||i==n){h[i][j]=1;if(x==n&&j>=y)h[i][j]=0;}
                    else
                        if(i==x&&j==y)h[i][j]=h[i+1][j];
                        else
                            h[i][j]=h[i+1][j]+h[i][j-1];
            if(h[1][m]<nrmin&&h[1][m]!=0)nrmin=h[1][m],xmin1=x,ymin1=y,nr=1;
            else
                if(h[1][m]==nrmin)nr++,xmin2=x,ymin2=y;}
    for(x=1; x<=n; x++)
        for(y=1; y<=m; y++)
        {
            for(i=n; i>=1; i--)
                for(j=1; j<=m; j++)
                    if(j==1||i==n){h[i][j]=1;if(y==1&&i<=x)h[i][j]=0;}
                    else
                        if(i==x&&j==y)h[i][j]=h[i][j-1];
                        else
                            h[i][j]=h[i+1][j]+h[i][j-1];
            if(h[1][m]<nrmin&&h[1][m]!=0)nrmin=h[1][m],xmin1=x,ymin1=y,nr=1;
            else
                if(h[1][m]==nrmin)nr++,xmin2=x,ymin2=y;}
    cout<<nrmin<<' '<<nr<<'\n';
    cout<<xmin1<<' '<<ymin1-1<<' '<<xmin1<<' '<<ymin1<<'\n';
    cout<<xmin2<<' '<<ymin2-1<<' '<<xmin2<<' '<<ymin2<<'\n';
    return 0;
}
```

```
4 6
21 2
1 5 1 6
4 1 4 2

Process re
Press any
```

The variable *nrmin* returns the minimum number of paths, and the pair of variables *(xmin1,ymin1)* and *(xmin2,ymin2)* carry the coordinates of the final nodes of the searched streets.

For *n=4* and *m=6*, we obtain that the minimum number of paths is *21*. There are *two* streets with this property, namely *the street that connects nodes* $(1,5)$ *and* $(1,6)$ and *the street that connects nodes* $(4,1)$ *and* $(4,2)$.

<u>Second method: A computing method</u>

The number of paths that *don't contain* a given street is minimal if and only if the number of paths *containing* that street is maximal. For each street *CD* of the map, we count how many paths contain that street. A path through *CD* is composed by a path from *A* to *C*, street *CD* and a path from *D* to *B*.

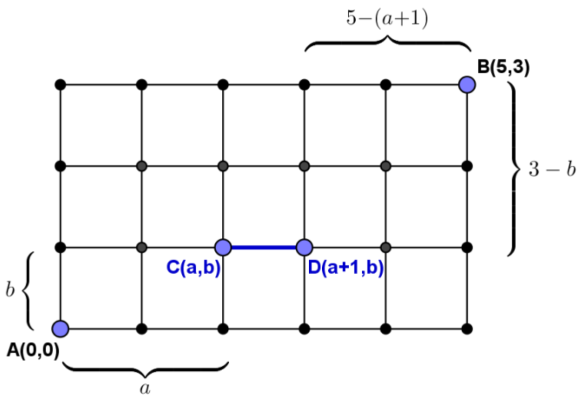


Figure 9

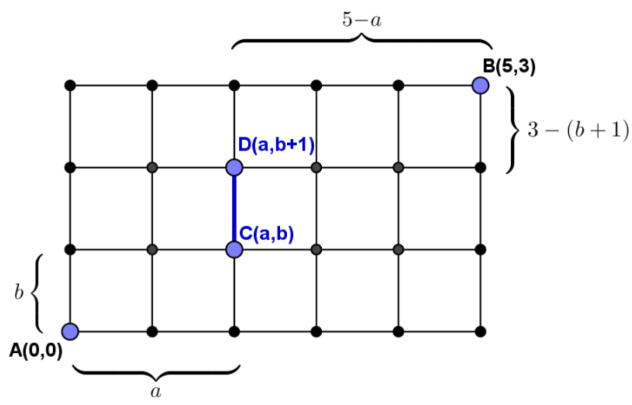


Figure 10

i) If the street is horizontal and joins point *C*, of coordinates (a,b), with point *D*, of coordinates $(a+1,b)$, where $0\le a\le 4$, $0\le b\le 3$ (see Figure 9), then the map of paths from *A* to *C* is a $a\times b$ rectangle and the map of paths from *D* to *B* is a rectangle with sides of length $5-(a+1)=4-a$ and $3-b$. Therefore, by *the product rule*, the number of paths that contain street *CD* is

$$D_1(a,b)=\binom{a+b}{a}\cdot\binom{(4-a)+(3-b)}{4-a}=\frac{(a+b)!}{a!\cdot b!}\cdot\frac{(7-a-b)!}{(4-a)!\cdot(3-b)!}.$$

ii) If the street is vertical and joins point *C*, of coordinates (a,b), with point *D*, of coordinates $(a,b+1)$, where $0\le a\le 5$, $0\le b\le 2$ (see Figure 10), than the map of paths from *A* to *C* is a $a\times b$ rectangle and the map of paths from *D* to *B* is a rectangle with sides of length $5-a$ and $3-(b+1)=2-b$. It follows that the number of paths that contain street *CD* is

$$D_2(a,b)=\binom{a+b}{a}\cdot\binom{(5-a)+(2-b)}{5-a}=\frac{(a+b)!}{a!\cdot b!}\cdot\frac{(7-a-b)!}{(5-a)!\cdot(2-b)!}.$$

We are now investigating the position and the orientation of the street *CD* for which the number of paths passing through this street is maximum. We have the following cases:

- If $b=0$, then

$$D_1(a,0)=1\cdot\frac{(7-a)!}{(4-a)!\cdot 3!}=\frac{(5-a)(6-a)(7-a)}{6};$$

$$D_2(a,0)=1\cdot\frac{(7-a)!}{(5-a)!\cdot 2!}=\frac{(6-a)(7-a)}{2}.$$

Because expressions $5-a, 6-a, 7-a$ get larger as *a* gets smaller, the maximum values in this case are $D_1(0,0)=35$ and $D_2(0,0)=21$; we keep in mind the highest value, $D_1(0,0)=35$

.

- If $b=1$, then

$$D_1(a,1)=\frac{(a+1)!}{a!\cdot 1!}\cdot\frac{(6-a)!}{(4-a)!\cdot 2!}=\frac{(a+1)(5-a)(6-a)}{2};$$

$$D_2(a,1)=\frac{(a+1)!}{a!\cdot 1!}\cdot\frac{(6-a)!}{(5-a)!\cdot 1!}=(a+1)(6-a).$$

From the table of values

a	0	1	2	3	4	5
$D_1(a,1)$	15	20	18	12	5	-
$D_2(a,1)$	6	10	12	12	10	6

we reason that in this case D_1 and D_2 take on values smaller than 35.

- If $b=2$, then

$$D_1(a,2)=\frac{(a+2)!}{a!\cdot 2!}\cdot\frac{(5-a)!}{(4-a)!\cdot 1!}=\frac{(a+1)(a+2)(5-a)}{2}$$

and the table of values

a	0	1	2	3	4
$D_1(a,2)$	5	12	18	20	15

shows, again, smaller values than 35. Moreover,

$$D_2(a,2)=\frac{(a+2)!}{a!\cdot 2!}\cdot 1=\frac{(a+1)(a+2)}{2},$$

and the expressions $a+1, a+2$ get larger as a gets larger, hence the highest value that we can obtain is $D_2(5,2)=21<35$.

- If $b=3$, then

$$D_1(a,3)=\frac{(a+3)!}{a!\cdot 3!}\cdot 1=\frac{(a+1)(a+2)(a+3)}{6},$$

and the expressions $a+1, a+2, a+3$ increase as a increases, so the maximum value in this case is $D_1(4,3)=35$.

In conclusion, the largest number of paths, 35, pass either through the street that joins the points of coordinates $(0,0)$ and $(1,0)$ or through the street that joins the points of coordinates $(4,3)$ and $(5,3)$ (the problem has *two* solutions – see Figure 11).

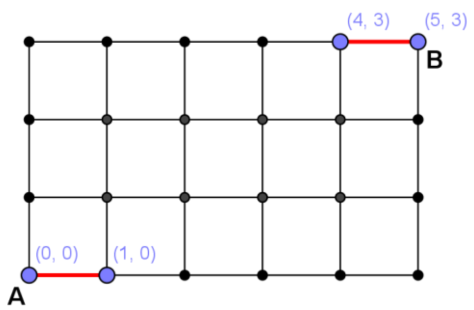


Figure 11

We observe that the street that joins the points of coordinates $(0,0)$ and $(1,0)$ is the street that joins the nodes $(4,1)$ and $(4,2)$ from the C++ program (see Figure 4). Also, the street that joins the points of coordinates $(4,3)$ and $(5,3)$ is the street that joins the nodes $(1,5)$ and $(1,6)$ from the program.

If one of these streets closes, the number of ways of driving from *A* to *B* is minimal. More precisely, since the total number of paths that join *A* and *B* is 56, there are $56-35=21$ possible paths left, as the C++ program also states.

On two of the *Big Hill* town streets (as marked in Figure 12) access is forbidden for cars and pedestrians alike.

(b) **In how many ways can one drive from point *A* to point *B* of *Big Hill*, taking into account the restrictions?**

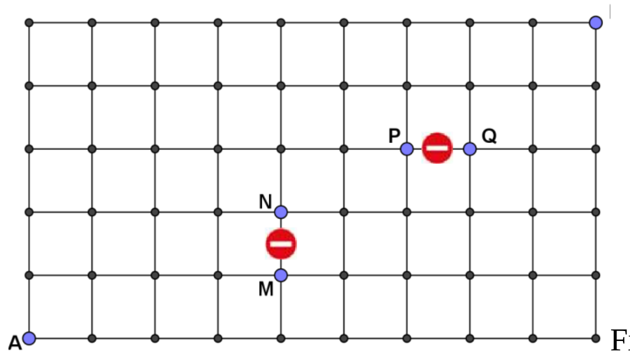


Figure 12. The centre of the town *Big Hill*

First, since the map in Figure 12 is a rectangle with the longer side of 9 streets and the shorter side of 5 streets, proceeding as at point (a), we infer that if none of the streets had been closed to traffic, the number of possible paths from *A* to *B* would have been

$$D_{AB}=\binom{9+5}{5}=\frac{14!}{9!\cdot 5!}=\frac{10\cdot 11\cdot 12\cdot 13\cdot 14}{1\cdot 2\cdot 3\cdot 4\cdot 5}=2002.$$

In order to calculate how many paths, avoid streets *MN* and *PQ*, we will subtract from D_{AB} the number D_{closed} of paths that contain at least one of the streets *MN* and *PQ*. According to *the inclusion-exclusion principle*, the value D_{closed} is the sum of the number D_{MN} of paths from

A to *B* that contain street *MN* and the number D_{PQ} of paths from *A* to *B* that contain street *PQ*, from which we subtract the number $D_{MN,PQ}$ of paths from *A* to *B* that contain both streets, because these paths have been summed twice.

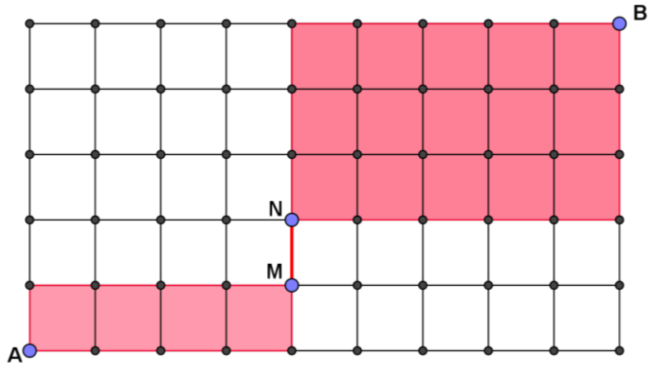


Figure 13

A path from *A* to *B* including street *MN* is the union of a path from *A* to *M*, street *MN* and a path from *N* to *B* (see Figure 13). Because the map of the streets from *A* to *M* is a 4×1 rectangle, there are $\binom{4+1}{1}=\binom{5}{1}=5$ possible paths *A*-*M*. Since the map of the streets from *N* to *B* is a 5×3 rectangle, there are $\binom{5+3}{5}=\frac{8!}{5!\cdot3!}=\frac{6\cdot7\cdot8}{6}=56$ admissible paths *N*-*B*. Therefore, $D_{MN}=5\cdot1\cdot56=280$.

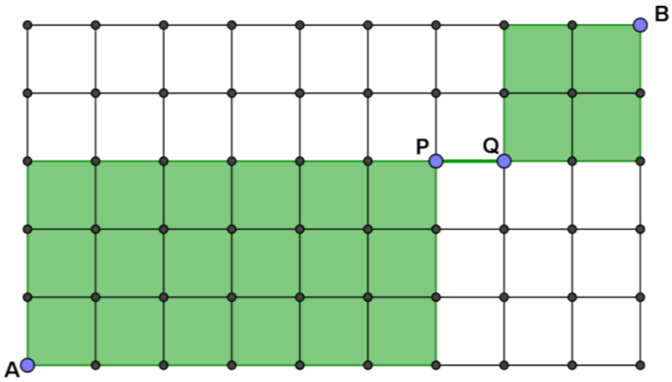


Figure 14

A path from *A* to *B* via street *PQ* is the union of a path from *A* to *P*, street *PQ* and a path from *Q* to *B*. As the map of streets from *A* to *P* is a 6×3 rectangle, and the map of streets from *Q* to *B* is a square with side length 2 (see Figure 14), we obtain analogously that

$$D_{PQ}=\binom{6+3}{6}\cdot1\cdot\binom{2+2}{2}=\binom{9}{6}\cdot\binom{4}{2}=\frac{9!}{6!\cdot3!}\cdot\frac{4!}{2!\cdot2!}=\frac{7\cdot8\cdot9}{6}\cdot\frac{3\cdot4}{2}=504.$$

Finally, a path from *A* to *B* containing streets *MN* and *PQ* is the union of a path from *A* to *M*, street *MN*, a path from *N* to *P*, street *PQ* and a path from *Q* to *B*. Because the map of the streets from *A* to *M* is a 4×1 rectangle, the map of the streets from *N* to *P* is a 2×1 rectangle, while the map of the streets from Q to B is a square with side length 2 (see Figure 15), we get

$$D_{MN,PQ} = \binom{4+1}{1}\cdot 1\cdot\binom{2+1}{1}\cdot 1\cdot\binom{2+2}{2} = \binom{5}{1}\cdot\binom{3}{1}\cdot\binom{4}{2} = 5\cdot 3\cdot\frac{4!}{2!\cdot 2!} = 15\cdot\frac{3\cdot 4}{2} = 90.$$

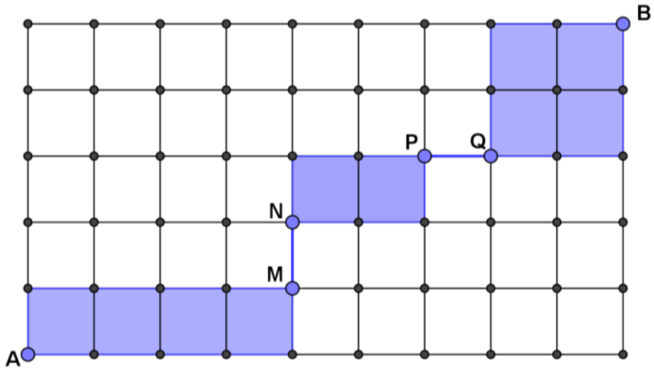


Figure 15

Consequently, $D_{closed} = D_{MN} + D_{PQ} - D_{MN,PQ} = 280+504-90 = 694$, hence the number of paths from A to B which avoid the restricted streets are $D_{AB} - D_{closed} = 2002-694 = 1308.$

(c) **Two walkers, *Alin* and *Bianca*, depart simultaneously from points *A* and *B*, respectively. *Alin* may move only one square right or upwards, while *Bianca* may move only one square left or downwards (when looking at the grid from the same perspective). For each of them, the choice of direction is made with equal probabilities in every intersection. What are the odds that *Alin* and *Bianca* meet?**

Because the map of the paths from A to B is a 9×5 rectangle, in order to get from A to B, no matter what path he may choose, *Alin* must walk 9 streets to the right and 5 streets upwards, that is $9+5=14$ streets. Likewise, in order to get from B to A, *Bianca* must also walk 14 streets (9 to the left and 5 downwards). **Supposing they walk at the same speed**, they must meet halfway, that is, after each of them walked $14:2=7$ streets.

Alin can walk the 7 streets in 2^7 equally possible ways, because from every node of the grid he has two equally possible ways of choosing the direction. Analogously, *Bianca* also has 2^7 equally possible ways of walking 7 streets. Hence, by *the product rule*, the pair *Alin-Bianca* may move along $2^7\cdot 2^7 = 2^{14}$ equally possible trajectories.

On the other hand, as they meet, we can merge their trajectories and we get a path that connects points A and B, travelling to the right or upwards. In other words, the set of the favorable cases is the set of all paths connecting A and B, on the conditions of the problem. The number of favorable cases is $\binom{14}{5} = 2002$, thus the probability that *Alin* and *Bianca* meet is equal to $\frac{2002}{2^{14}} = \frac{1001}{2^{13}} = \frac{1001}{8192} = 0{,}122$.

(d) **Apart from the two above mentioned restrictions, other streets in the town will be under repair and will be inaccessible for a while. Find out the maximum number of streets that can be under repair simultaneously, so that all the touristic objectives are connected to point *A* by at least one path. We maintain the original rules of movement. The path starting at *A* may end in any point (touristic objective).**

We count how many streets we may close if we want to reach from *A* to any other node, not necessarily getting to *B*.

Node N_1, the bottom right corner of the rectangle, can be reached only by following the bottom edge of the map, hence the streets on this edge cannot be under repair. At its turn, node N_2, the upper-left corner of the rectangle, can be reached only by following the left vertical edge of the map, hence the streets on this edge cannot be under repair either (see Figure 16). In this way, we obtain that *every* node on these two edges can be reached by a path starting at *A*.

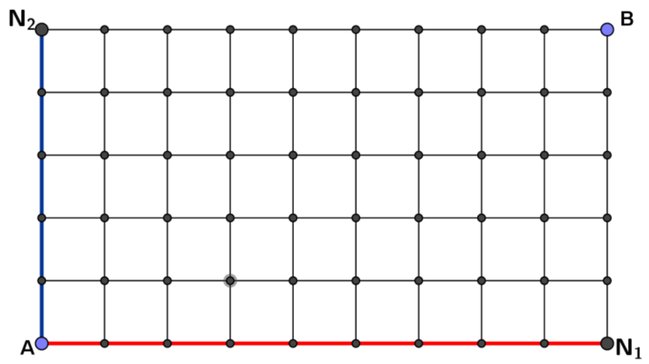


Figure 16

The rest of the nodes on the map can be reached from two directions (namely, from the left and from below), hence if we keep one of these streets, we may have at least a path from *A* to each of these nodes. Since the grid has 10 nodes on the horizontal and 6 nodes on the vertical, by removing the $10+6-1=15$ nodes on the left and bottom sides, there are $6\cdot 10-15=45$ nodes left. In conclusion, in this case we may close at most 45 *streets*.

An example of streets closure is given in Figure 17.

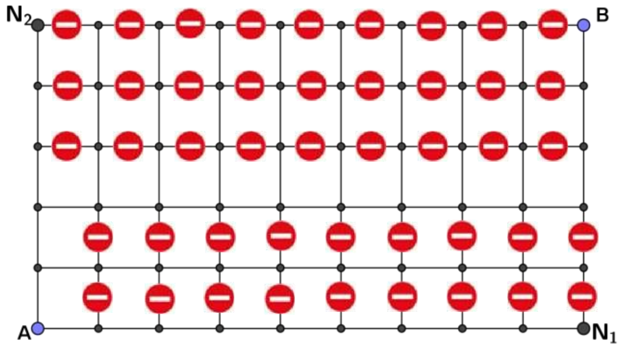


Figure 17

Conclusions

We have answered all the questions completely. We have also added supplementary tasks to the problem, such as the C++ programming and different approaches to the solution. By studying this problem, we have learned more techniques in combinatorics, probability theory and their applications. The problem could be a starting point in solving more delicate problems, such as the traffic light coordination problem in busy city centres.

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1. J. Batterson, *Competition Math For Middle School*, AoPS Inc., 2011.
2. D. Fomin, S. Genkin, I. Itenberg, *Mathematical Circles (Russian Experience)*, Universities Press, 1996.

This article is written by students. It may include omissions and imperfections.

Probabilities and Triangles

2016- 2017

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Presentation of the research topic

1. Assume a stick is randomly broken into three pieces. What is the probability that the obtained pieces can form a triangle?
2. Let be the following numbers: 1, 2, 3, 4,…., n. We randomly choose 3 of them. What is the probability that the chosen numbers can be the lengths of the sides of a triangle?

Brief presentation of the conjectures and results obtained

1st problem

A stick is broken into three pieces. What is the probability that the obtained pieces can form a triangle?

First, we used the following algorithm on computer to determine the probability for the first problem and the result was approximately 0.25.

```
#include <iostream>
using namespace std;
int l,a,b,c;
float nt,nf;
int main()
{
    cout<<"l=";
    cin>>l;
    for (a=1;a<=l-1;a++)
    {
        for (b=1;b<=l-a;b++)
        {
          c=l-a-b;
          if (a<b+c && b<c+a && c<a+b)
            nf++;
         if (c>0)
            nt++;
        }
```

```
    }
    cout<<nf<<" "<<nt<<endl;
    cout<<nf/nt;
    return 0;
}
```

Then we followed another idea. Let’s take an equilateral triangle with the height equal to the stick’s length. From a point P inside the triangle we drop perpendiculars on the triangle’s sides.

According to **Vivianni’s theorem**: *“In an equilateral the sum of the perpendicular lines dropped from a point P inside the triangle on the triangle’s sides is equal to the height of the triangle”.*

This is a proof to Vivianni’s theorem:
We drop the perpendiculars PX, PY, and PZ from P on the sides of the triangle. We have to demonstrate that: PX+PY+PZ=AD, where AD represents the height of the equilateral triangle.
Let S represent the area of the triangles. As we can see in the picture:
$S(\Delta PBC) + S(\Delta PAB) + S(\Delta PCA)=S(\Delta ABC)$
$\Leftrightarrow \frac{PX*BC}{2}+\frac{PY*AB}{2}+\frac{PZ*AC}{2}=\frac{AD*l}{2}$
But AB=BC=CA=l $=> \frac{PX*BC}{2}+\frac{PY*l}{2}+\frac{PZ*l}{2}=\frac{AD*l}{2} \Leftrightarrow PX+PY+PZ=AD$

Back to our problem:
Now we have to find the probability that an arbitrarily chosen point P lying inside ΔABC satisfies the relations: PX+PY>PZ; PY+PZ>PX; PZ+PX>PY.

Solution:

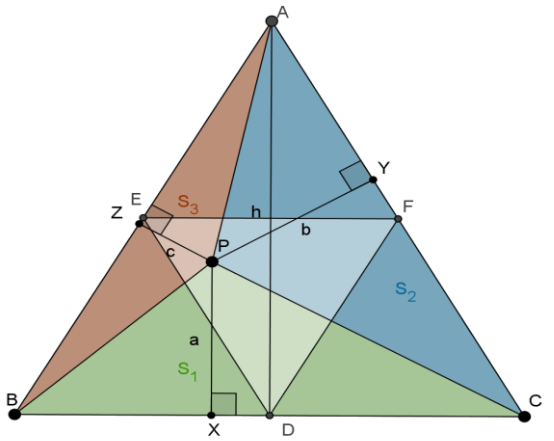

Figure 1. Representing Vivianni’s theorem

$S(\Delta BCP) =\frac{l*PX}{2} => PX=\frac{2*S(\Delta BCP)}{l}$
$S(\Delta ABP) =\frac{l*PY}{2} => PY=\frac{2*S(\Delta ABP)}{l}$
$S(\Delta CAP) =\frac{l*PZ}{2} => PZ=\frac{2*S(\Delta CAP)}{l}$
But PX<PY+PZ
$\Leftrightarrow \frac{2*S(\Delta BCP)}{l} < \frac{2*S(\Delta ABP)}{l}+\frac{2*S(\Delta CAP)}{l}$

⇨ S(ΔBCP) < S(ΔABP) + S(ΔCAP)

S(ΔABP) + S(ΔCAP) + S(ΔBCP) = S(ΔABC)

⇨ S(ΔBCP) < S(ΔABC) - S(ΔBCP)

⇨ S(ΔBCP) $<\frac{S(\Delta\text{ABC})}{2}$ => $\frac{PX*l}{2}<\frac{AD*l}{4}$

AD=h => PX$<\frac{h}{2}$.

Similarly, we can prove that PY$<\frac{h}{2}$ and PZ$<\frac{h}{2}$.

The middle points of the heights in a triangle belong to the middle lines of the triangle. So point P must be inside ΔDEF, where D, E, F are the middle points of (BC), (AB) and (CA).

We can conclude:

- The favourable cases are for point P inside ΔDEF.
- The possible cases are for point P inside triangles ΔAEF, ΔBDE, ΔCDF or ΔDEF.

Because these four triangles are equivalent, point P has 25% chance of lying within the triangle DEF. This is the answer for the first problem.

2nd problem

We randomly choose 3 numbers that are smaller than or equal to n. What is the probability that these numbers can represent a triangle's sides?

We can find out the number of the possible cases using combinations of n elements in groups of 3:

$$C_n^3 = \frac{n!}{(n-3)!*3!} = \frac{(n-2)(n-1)n}{6}$$

First, we elaborated, like with the precedent problem, an algorithm:

```
#include <iostream>
using namespace std;
int n,a,b,c;
float nt,nf;
int main()
{
   cout<<"n=";
   cin>>n;
   for (a=1;a<=n-2;a++)
   {
      for (b=a+1;b<=n-1;b++)
      {
        for (c=b+1;c<=n;c++)
        {
           if (a<b+c && b<c+a && c<a+b)
           {
              nf=nf+1;
              cout<<a<<" "<<b<<" "<<c<<endl;
           }
           nt++;
```

```
        }

      }
   }
   cout<<"favourable cases "<<nf<<endl;
   cout<<"possible cases "<<nt<<endl;
   cout<<"probability is "<< nf/nt;
   return 0;
}
```

For the set {1,2, …, 8} we ran this application and obtained the following results: the number of the favourable cases was 22, the number of possible cases was 56, so the probability turned out to be 0.392857.

Then, we ran it for the second time, now for a different set: {1,2, 3,…,100}. The results were: 79,625 favourable cases, 161,700 possible cases, and the probability was 0.492424.

Conclusions

1st problem: If we break a stick into three pieces, the probability that the pieces can form a triangle is exactly 25%.
2nd problem: For now, we have only managed to solve the problem using the CodeBlocks application, but we are close to finding the mathematical solution to it as well.

This article is written by students. It may include omissions and imperfections.

Pyramids

2016-2017

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Presentation of the research topic

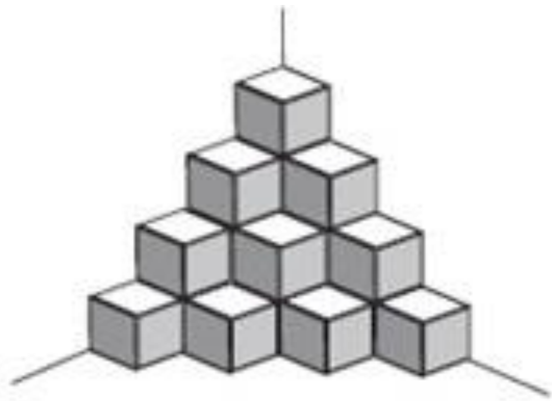
The tower represented in this image is built of matching cubes, of side 1, stacked one over the other and glued to the corner of a wall. Some of these cubes are not visible from this position/perspective.

a) How many cubes will a tower of height of 30 have?

b) A number $n \geq 3$ of cubes, placed side by side covers perfectly a square. For what values of n can we rearrange the cubes so that we can build a pyramid as the one in the image, without remaining any unused cubes? For every found value of n, what height has the built pyramid?

c) Build a regular triangular pyramid by overlapping some spheres of diameter 1 (instead of cubes). What is the height of such a pyramid formed with 1330 balls?

d) What is the volume of the minimal tetrahedron in which the pyramid found at point c) can be inscribed?



Brief presentation of the conjectures and results obtained

Part a)

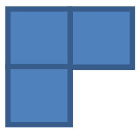
How many cubes will a tower of height of 30 have?

Solution 1

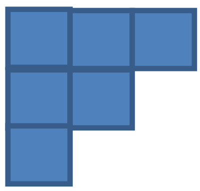
In the first part, we found a rule of how is the figure made.

- In the first layer, seen from the top, we see a single cube.

- In the second layer, we see $(1 + 2 = 3)$ cubes.



- In the third layer, we see $(1 + 2 + 3 = 6)$ cubes.



- And so on…
- So we find a rule of the number of the cubes that are in a layer.

We found after that a rule of the number of the cubes that are in a layer.

This equation is

$$n = 1 + 2 + 3 + \cdots + k$$

$$n = \frac{(1+k)\cdot k}{2}$$

where k is the number of the layer and n is the number of cubes in that layer.

By using that rule, we find the final number of cubes for a pyramid of height 30, that is 4960.

Solution 2

We denote by C_n the number of cubes that are in a tower having $n \geq 1$ levels.

We can easily observe that:

$$C_1 = 1,\ C_2 = 4,\ C_3 = 10,\ C_4 = 20, C_5 = 35, C_6 = 56.$$

We are looking for a recurrent relation involving C_n. Firstly, let us denote by $d_n = C_{n+1} - C_n$, that is the difference of the number of cubes between consecutive levels. We observe that

$$d_n = d_{n-1} + n + 1, \text{ for all } n \geq 2.$$

Coming back to C_n, we find the recurrence

$$C_{n+1} - C_n = C_n - C_{n-1} + n+1, \text{ for all } n \geq 2,$$

which is equivalent to

$$C_{n+1} = 2C_n - C_{n-1} + n+1, \text{ for all } n \geq 2,$$

By giving successively values to n, we find that $C_{30} = 4960$.

Solution 3

Below, we write the differences, the difference of differences and so on, for the consecutive values of C_n.

1 4 10 20 35 56

3 6 10 15 21

3 4 5 6

1 1 1

We observe that after three iterations we arrive at a constant sequence. Therefore, the general term of the sequence will be of the form: $C_n = a\,n^3 + b\,n^2 + cn + d$. By writing the first four terms of the sequence, we have:

$$C_1 = 1 = a + b + c + d$$

$$C_2 = 4 = 8a + 4b + 2c + d$$

$$C_3 = 10 = 27a + 9b + 3c + d$$

$$C_4 = 20 = 64a + 16b + 4c + d.$$

By solving the system for $a, b, c, d,$ we get:

$$a = \frac{1}{6};\ b = \frac{1}{2};\ c = \frac{1}{3}\ ;\ d = 0.$$

And thus,

$$C_n = \frac{1}{6}n(n + 1)(n + 2) = \binom{n+2}{3},$$

with C_{30}= 4960.

Part b)

A number $n \geq 3$ of cubes, placed side by side covers perfectly a square. For what values of n can we rearrange the cubes so that we can build a pyramid as the one in the image, without remaining any unused cubes? For every found value of n, what height has the built pyramid?

We found a rule for the number of cubes used to build a pyramid of height h.

n cubes should be put in a square, so *n* should be a perfect square.

We observed that the formula that we made is the formula of the tetrahedral numbers. So, *n* should be a tetrahedral number.

But the only tetrahedral numbers that are making square numbers are 1,4 and 48.

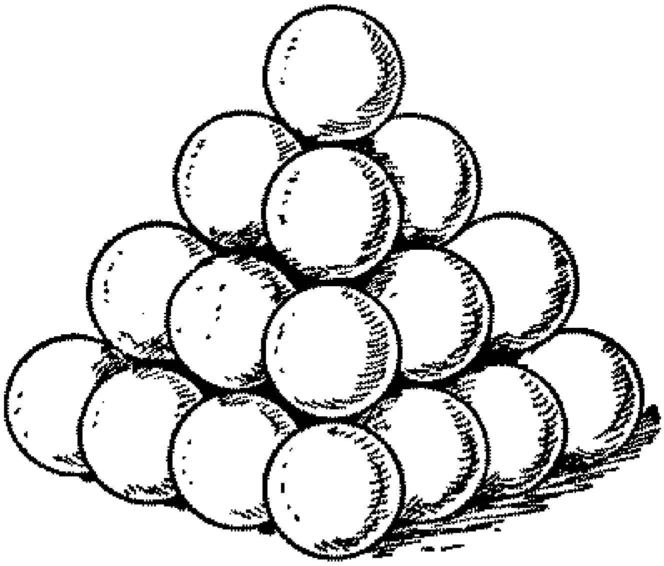
If the height is 1, the *n* is 1 and *l* is 1, where *l* is the side of the square covered by that *n* cubes.

If the height is 2, the *n* is 4 and *l* is 2.

If the height is 48, the *n* is 19600 and *l* is 140.

Part c)

Build a regular triangular pyramid by overlapping some spheres of diameter 1 (instead of cubes). What is the height of such a pyramid formed with 1330 balls?



Using the rule that we found at part a), we discovered that the required pyramid has 19 layers. Indeed, by solving in $\boldsymbol{N}$ the equation

$$\frac{1}{6}n(n+1)(n+2) = 1330,$$

We find that $n = 19$.

As discussed before, the number of balls on the bottom layer is $\frac{n(n+1)}{2}$

Solving in $\boldsymbol{N}$ the equation

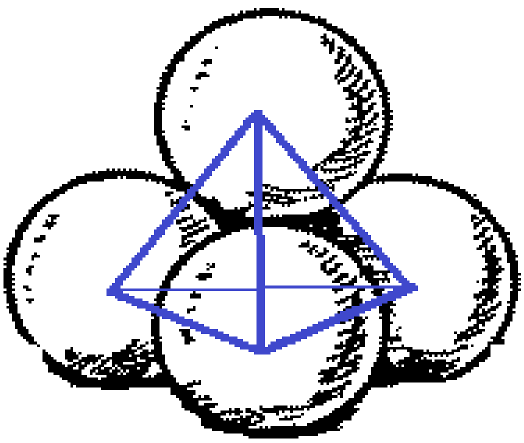
$$\frac{n(n+1)}{2} = 190,$$

We obtain $n = 19$. Therefore, the base length of a pyramid having 190 balls is 19.

The height of the pyramid formed by stacking spheres is a function of *n* and *d* (the diameter of each sphere). We need to find the height of the pyramid in terms of *n* and *d*.

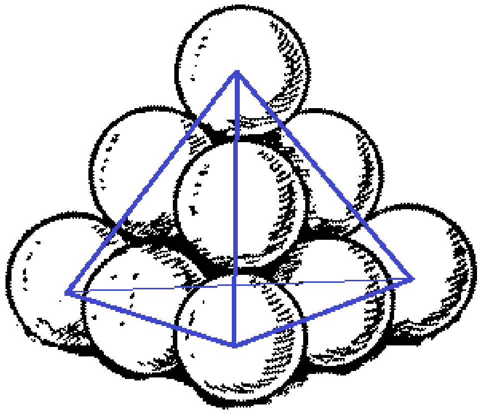
Firstly, we consider the case of a tetrahedron made from four balls: three balls on the bottom and one ball at the top. The centres of these spheres form a tetrahedron with an edge length of *d* and a height of $d\sqrt{\frac{2}{3}}$. The total height of stack of spheres is

$$d + d\sqrt{\frac{2}{3}}.$$



We now add a third layer to the bottom of the pyramid (add six balls) and obtain a total of 10 balls. We then consider the tetrahedron formed by the centres of the spheres at the corners of the pyramid. This tetrahedron has an edge length of $2d$ and a height of $2d\sqrt{\frac{2}{3}}$.
The total height of the stack is

$$d + 2d\sqrt{\frac{2}{3}}.$$



In general, the total height of a pyramid with n levels is

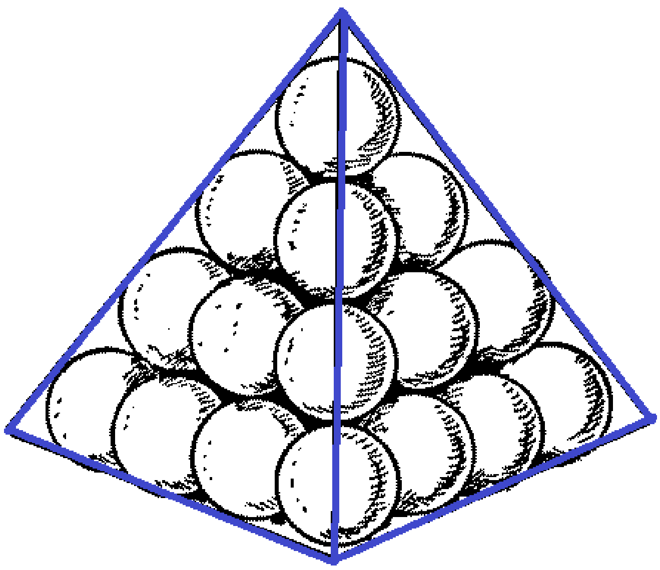
$$d + (n-1)d\sqrt{\frac{2}{3}}.$$

For $n = 19$ and $d = 1cm$, we get that the height of a pyramid made of 1330 similar balls is

$$h \cong 15.7\ cm.$$

Part d)

What is the volume of the minimal tetrahedron in which the pyramid found at point c) can be inscribed? What fraction of the space does the ball pyramid occupies inside the tetrahedron?

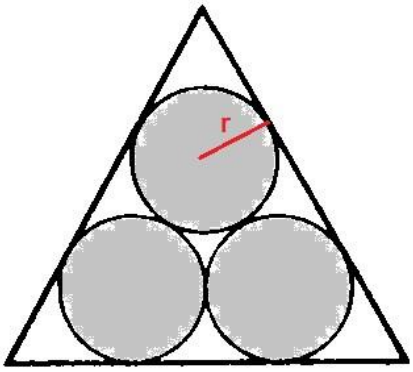


The base of the 2 level pyramid can be inscribed in an equilateral triangle with a side length of

$$d + d\sqrt{3}.$$

In general, the base of the n level pyramid can be inscribed in an equilateral triangle with a side length of

$$d(n-1) + d\sqrt{3}.$$



For $n = 19$ and $d = 1cm$, we obtain that the side length of a regular pyramid is $l = 18 + \sqrt{3}$. The volume of the minimal tetrahedron is

$$V = l^3 \frac{\sqrt{2}}{12} \cong 905.42\ cm^3$$

The volume of a single sphere is $V_{sphere} = \frac{4\pi R^3}{3} = \frac{\pi d^3}{6} = \frac{\pi}{6}$, and so, the volume of the pyramid made of 1330 spheres is

$$V_p = 665 \frac{\pi}{3} \cong 696.39\ cm^3,$$

representing a fraction of about 77% out of the volume of the tetrahedron.

Our work

Part a)

In the first part, we found a rule of how is the figure made.

In the first layer, seen from the top, we see a single cube.

In the second layer, we see (1+2=3) cubes.

In the third layer, we see (1+2+3=6) cubes.

And so on…

So, we find a rule of the number of the cubes that are in a layer.

This equation is (with *k* the number of the layer and *n* the number of cubes in that layer):

$$n = 1 + 2 + 3 + \cdots + k$$

$$n = \frac{(1+k) \cdot k}{2}$$

So, using that equation, we found that the total number of cubes that are needed to make a tower of height 30 is:

$$n = \frac{1}{2}[(1^2 + 2^2 + 3^2 + \cdots + 30^2) + (1 + 2 + 3 + \cdots + 30)]$$

$$n = \frac{1}{2}\left[(1^2 + 2^2 + 3^2 + \cdots + 30^2) + \frac{30 \cdot 31}{2}\right]$$

But, $1^2 + 2^2 + 3^2 + \cdots + \mathrm{n}^2 = \frac{\mathrm{n}(\mathrm{n}+1)(2\mathrm{n}+1)}{6}$.

So, the final number of cubes for a pyramid of height 30 is:

$$n = \frac{1}{2} \cdot \left[\frac{30 \cdot 31 \cdot 61}{6} + \frac{30 \cdot 31}{2}\right] = \frac{1}{2} \cdot (9455 + 465)$$

$$n = 4960$$

Part b)

Let's say $\mathrm{C_n}$ is the number of cubes in a pyramid of height *n*.

If *n*=1, then $\mathrm{C_n}$=1

If *n*=2, then $\mathrm{C_2}$=4

If *n*=3, then $\mathrm{C_3}$=10

If *n*=4, then $\mathrm{C_4}$=20

If *n*=5, then $\mathrm{C_5}$=35

If *n*=6, then $\mathrm{C_6}$=56

So, we found a rule for the number of cubes used to build a pyramid of height h.

$$n = \frac{1}{2}\left[\frac{\mathrm{h}(\mathrm{h}+1)(2\mathrm{h}+1)}{6} + \frac{\mathrm{h}(\mathrm{h}+1)}{2}\right] = \frac{\mathrm{h}(\mathrm{h}+1)(\mathrm{h}+2)}{6}$$

But, *n* cubes should be put in a square, so *n* should be a perfect square.

We observed that this is the formula of the tetrahedral numbers. So *n* should be a tetrahedral number.

A ***tetrahedral number*** is a figurate number that represents a pyramid with a triangular base and three sides.

But the only tetrahedral numbers that are making square numbers are 1,4 and 48.

If the height is 1, the *n* is 1 and *l* is 1, where *l* is the side of the square covered by that *n* cubes.

If the height is 2, the *n* is 4 and *l* is 2.

If the height is 48, the *n* is 19600 and *l* is 140.

But *n* must be equal or bigger than 3, so *n* can take the values 4 or 48.

Part c)

Using the rule that we found at a):

$$1330 = \frac{1}{2}[(1^2 + 2^2 + 3^2 + \cdots + n^2) + (1 + 2 + 3 + \cdots + n)]$$

$$1330 = \frac{1}{2}\left[\frac{\mathrm{n(n+1)(2n+1)}}{6} + \frac{\mathrm{n(n+1)}}{2}\right]$$

Where h is the height required.

$$2660 = \frac{\mathrm{n(n+1)(2n+1)}}{6} + \frac{\mathrm{n(n+1)}}{2}$$

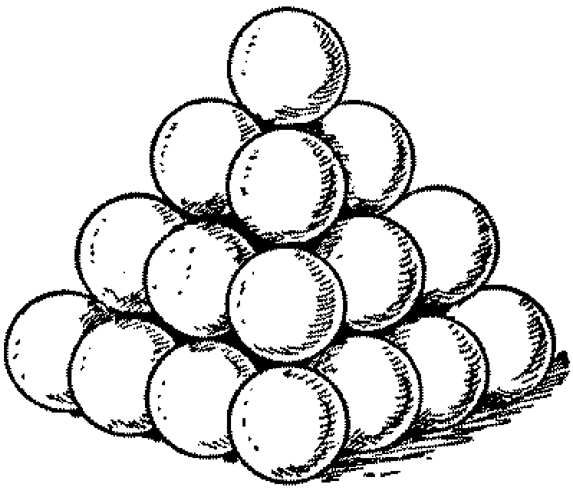
$$2660 = \frac{\mathrm{n(n+1)(2n+1) + 3n(n+1)}}{6}$$

$$2660 = \frac{n(n+1)(2n+1+3)}{6} = \frac{n(n+1)(n+2)}{3}$$

$$n(n+1)(n+2) = 7980$$

So, by giving values we found that the height of the pyramid is 19.

The second possibility is the following one:



Using the rule that we found at part a), we discovered that the required pyramid has 19 layers. Indeed, by solving in $\boldsymbol{N}$ the equation

$$\frac{1}{6}n(n\ +\ 1)(n\ +\ 2) = 1330$$

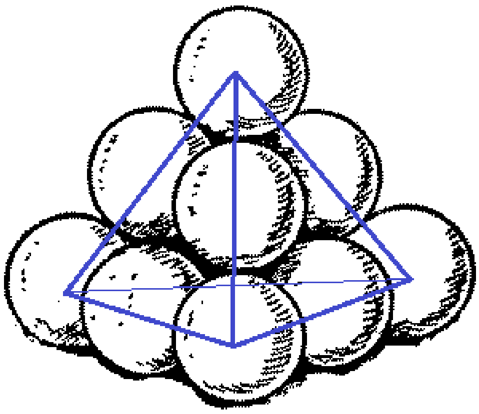
We find that $n = 19$.

As discussed before, the number of balls on the bottom layer is $\frac{n(n+1)}{2}$.

Solving in $\boldsymbol{N}$ the equation $\frac{n(n+1)}{2} = 190$,

We obtain $n = 19$. Therefore, the base length of a pyramid having 190 balls is 19.

The height of the pyramid formed by stacking spheres is a function of n and d (the diameter of each sphere). We need to find the height of the pyramid in terms of n and d.



In general, the total height of a pyramid with n levels is

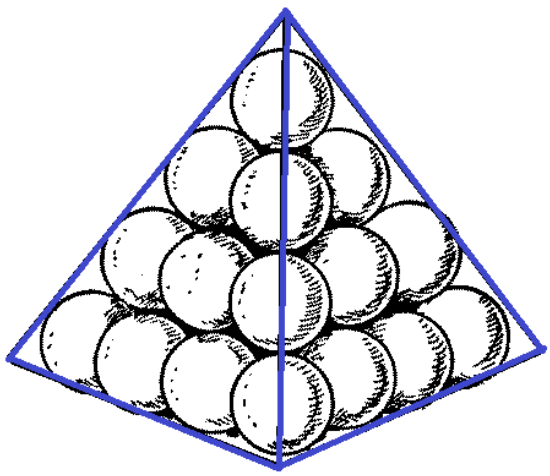
$$d + (n-1)d\sqrt{\frac{2}{3}}.$$

For $n = 19$ and $d = 1cm$, we get that the height of a pyramid made of 1330 similar balls is

$$h \cong 15.7\ cm.$$

Part d)

What is the volume of the minimal tetrahedron in which the pyramid found at point c) can be inscribed? What fraction of the space does the ball pyramid occupies inside the tetrahedron?



In general, the base of the n level pyramid can be inscribed in an equilateral triangle with a side length of

$$d(n-1) + d\sqrt{3}.$$

For $n = 19$ and $d = 1cm$, we obtain that the side length of a regular pyramid is $l = 18 + \sqrt{3}$.

The volume of the minimal tetrahedron is

$$V = l^3 \frac{\sqrt{2}}{12} \cong 905.42\ cm^3$$

The volume of a single sphere is $V_{sphere} = \frac{4\pi R^3}{3} = \frac{\pi d^3}{6} = \frac{\pi}{6}$, and so, the volume of the pyramid made of 1330 spheres is

$$V_p = 665\frac{\pi}{3} \cong 696.39\ cm^3,$$

representing a fraction of about 77% out of the volume of the tetrahedron.

Conclusions

In conclusion, we proved in the part a) that there were needed 4960 cubes to make a pyramid of height 30.

We also proved in part b) that if we make a pyramid of height 4 or 48, we can rearrange the cubes so they can make a square.

In c) we proved that we need 19 levels to have 1330 spheres in a pyramid, then we calculated the height of a pyramid as the one above: for $n = 19$ and $d = 1cm$, we get that the height of a pyramid made of 1330 similar balls is

$$h \cong 15.7\ cm.$$

At the end, we proved that for $n = 19$ and $d = 1cm$, we obtain that the side length of a regular pyramid is $l = 18 + \sqrt{3}$. The volume of the minimal tetrahedron is

$$V = l^3 \frac{\sqrt{2}}{12} \cong 905.42\ cm^3$$

The volume of a single sphere is $V_{sphere} = \frac{4\pi R^3}{3} = \frac{\pi d^3}{6} = \frac{\pi}{6}$, and so, the volume of the pyramid made of 1330 spheres is

$$V_p = 665\frac{\pi}{3} \cong 696.39\ cm^3,$$

representing a fraction of about 77% out of the volume of the tetrahedron.

Reference

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This article is written by students. It may include omissions and imperfections.

Taxi Geometry

2016- 2017

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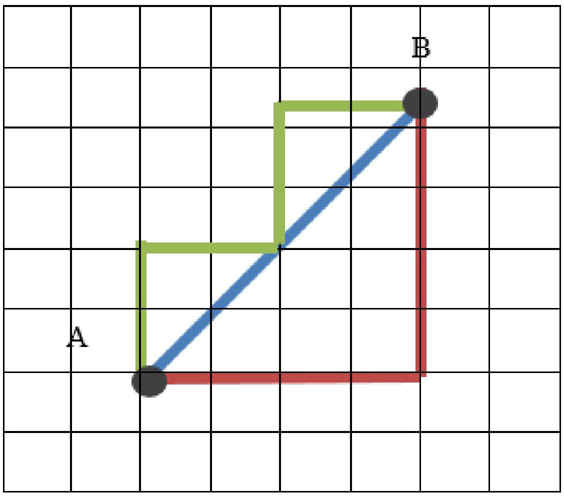
1. Presentation of the research topic

Geometry has been known to humanity ever since Euclid had first thought to study the science of shapes and lines, giving us the basics that students around the globe have been taught for hundreds of years now. Its simplicity is only completed by its paradoxical complexity as many mind-boggling problems arose during the passing of the ages. To begin them all, we have to think of the basic notions of **Euclidian geometry**, as it’s called. We have points everywhere, and we can connect them all using lines. For the sake of the presentation we are going to assume that the lines we use to make these ties are only **straight,** because there are an infinite number of curves we can use for the same purpose. Given this fact, we are certain that between two points there can **only be one** such line that we call **distance**. The emphasis falls on the word one, which in terms defines the uniqueness of the distance. However, this is true only for the way Euclid taught us to reason. We can apply this in a multitude of domains, but there are some places we just can’t make do with the ancient lessons of the famous Greek scientist. For example, we cannot calculate the distance between a pizza parlour and the home of one of their customer just like that, because the poor delivery boy can’t pass through buildings, instead he has to go, like every other human being, on the road. Taking the perpendicular streets of Manhattan as a working ground, we can notice an interesting approach to calculating a distance.

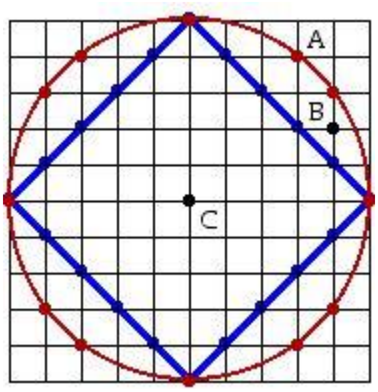


$d_E(A,B)=\sqrt{x(_A - x_B)^2 + (y_A - y_B)^2}$ **(1.1)**

$d_T(A,B)=|x_A - x_B| + |y_A - y_B|$ **(1.2)**

This would be the formula of the distance in the Euclidian Geometry, the line itself being highlighted in dark blue **(1.1)**. However, if we were to calculate it from the perspective of our delivery boy, it would be the red one. Thus, we can consider that the distance the underpaid employee must travel is given by the length of the two red lines **(1.2)**. However, we can also notice that it is not necessary for us to go on the same road time and time again; following the green lines will result in the same length as before, only in a different way. To conclude, we can notice that the distance between two points in what we call "Taxi Geometry" is NOT unique. The key difference that we remark and that separates the Euclidian Geometry from the Taxi Geometry is the formula for the distance, but this, however brings forth a completely different array of complications, many of which will be covered in this paper.

2. The representation of the circle



In the Euclidian geometry, the circle is defined as the "geometrical locus of points, equally distanced from a fixed point, named origin or centre". How does this translate in *Taxi Geometry*? We will take the origin of the cartesian system as the centre of our circle, and the radius equal to one. Using the aforementioned formula for the distance, we can conclude that the representation of a circle in *Taxi Geometry* is, in fact, a square.

3. The number pi (π)

Since we had proven that a circle translates to a square, in *Taxi Geometry,* how does that affect one of the most famous mathematical constants there is, namely pi? First of all, how do we calculate pi? Euclid has given us the first way of approximating this transcendental number, by simply making a circle out of a rope, measuring it, then dividing its length with that of the diameter that he had represented in the same way, using a piece of string. Therefore, we have the formula for pi:

$$\boldsymbol{\pi = \frac{C}{2r}}$$

Since our 'circle' is now a square, its circumference is its perimeter, and its 'radius' is the same one that we have chosen, that is 1. Using the reasoning behind the distance in *Taxi Geometry* we can see that the length of one side of the square is 2.

$$\boldsymbol{\pi = \frac{4l}{2r} = \frac{8}{2} = 4}$$

Thus, the irrational number pi, with its famous infinite decimal places has become nothing more, but the bland integer, 4.

RIP π (1737-1952)

4. The mediator of a segment

To find the mediator of a segment in *Taxi Geometry*, first of all we need to calculate the middle of said segment.
Considering the segment AB with the coordinates A (3, 2) and B (5, 4), an arbitrary point on the mediator M (α, β) and a point P, whose coordinates we need to find.

$$dt_{(MA)} = dt_{(MB)} \Rightarrow |\alpha - 3| + |\beta - 2| = |\alpha - 5| + |\beta - 4|$$

Considering α<3, we have:

$$3 + \alpha + |\beta - 2| = 5 - \alpha + |\beta - 4| \Rightarrow |\beta - 2| = |\beta - 4| + 2$$

By analyzing all the possible cases for β we obtain

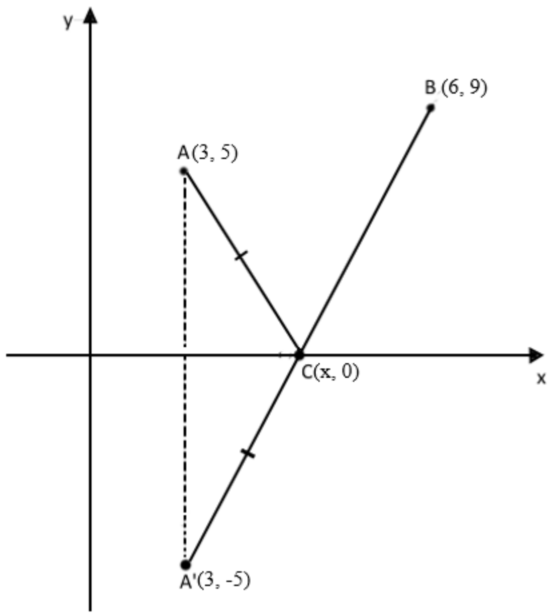
$\forall \beta \in [4, \infty)$, so $P(\alpha,\beta)$ with α <3 and $\beta \in [4, \infty)$

Considering that:

$$\alpha \in [3,5] \Rightarrow \alpha - 3 + |\beta - 2| = 5 - \alpha + |\beta - 4| \Rightarrow |\beta - 2| = |\beta - 4| + 8 - 2\alpha$$

We obtain that P(5,β), with $\beta\beta \in (-\infty, 2]$
For $\beta \geq 4 \Rightarrow \alpha = 5$ and, finally, for $\alpha > 5$, the equation of the mediator is
$\alpha - 3 + |\beta - 2| = \alpha - 5 + |\beta - 4| \Rightarrow |\beta - 2| = |\beta - 4| - 2$,with the solution $\beta \in (-\infty, 2]$
We can see that this result doesn't resemble the Euclidian Geometry result, at all, however, if the segment is parallel to any of the axis of the cartesian system, the calculations for the mediator are the same, be them in Taxi or Euclidian Geometry.

5. The billiard problem (A problem of space efficiency)



Let A and B be two different cities, the distance between them irrelevant, and C the location of a gas station. We can move the position of the gas station, as it has not yet been built, and we strive for perfection and efficiency. The two cities are major industrial centres, therefore trucks, lorries and buses leaving every day to link them. However, since the road is long and no one vehicle can go on that distance without enough fuel, the county administration decides to build a gas station…on the freeway near which no city is situated… This means that everyone will have to go from A to C then from C to B in order to travel between cities. However, since the administration demands efficiency, this distance is supposed to be minimum.

The solution in Euclidian geometry is easy, because of the uniqueness of the shortest road between two points. What we have to do is to consider the freeway as the X axis, and then create a "ghost town A" across it, creating its symmetrical point. Then, by intersecting the

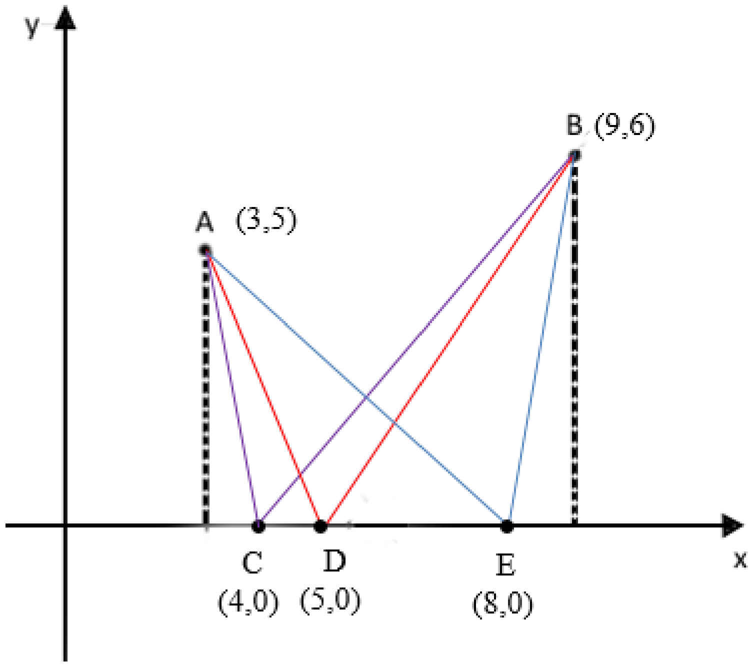
road between the ghost town and our ghost town A (A' for reference) and B with the X-axis we can find out the solution. The shortest distance between A and B through C is going to be the straight line from A' to B, because, due to laws of symmetry, AC=A'C. However, things change with the use of Taxi Geometry.
Let A(3,5) and B(9,6) be our points, for the sake of the explanation. The distances are expressed as:

AC=|3-x|+|5-0|
BC=|9-x|+|6-0|, therefore
AB+BC=|3-x|+|9-x|+11, minimum

We have to solve for x, therefore let f(x)=|3-x|+|9-x|. We have the following definition for our function:

$$f(x) = \begin{cases} 12 - 2x, x < 3 \\ 6, x \in [3,9] \\ 2x - 12, x > 9 \end{cases}$$

By graphing the function, we can notice that the minimum is not in one point, but rather one interval of numbers, in our case [3,9]. Therefore, we can determine there is an infinity of values for x that can solve our equation. Whereas in the Euclidian Geometry, we only had one point for our gas station, now we have an entire segment where we can place our much-needed pump, along with a lovely shopping centre.

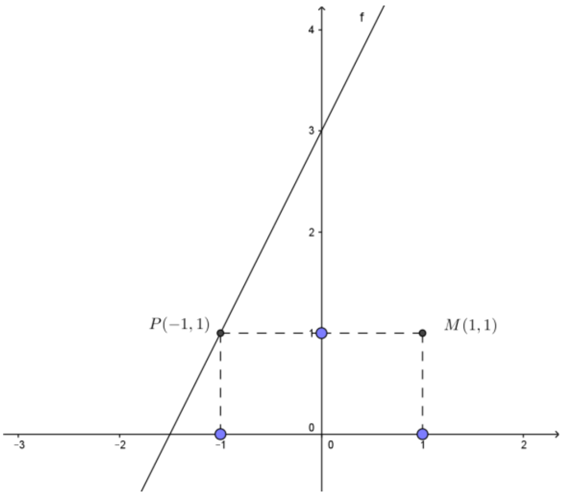


AC+CB=AD+DB=AE+BE
|3-4|+|5-0|+|4-9|+|6-0|=|3-5|+|5-0|+|5-9|+|6-0|=|3-8|+|5-0|+|8-9|+|6-0|
1+5+5+6=2+5+4+6=5+5+1+9
17=17=17

6. Calculating the distance between a point and a line in TG

In the Euclidian Geometry, the distance from the point $M(X_0, Y_0)$ to the line $ax+by+c=0$ is the shortest way from the point to the line and its calculated using the formula:

$$d = \frac{|aX_0 + bY_0 + c|}{\sqrt{a^2 + b^2}}$$



Let's consider a numerical example: the distance from $M(1,1)$ to the line $2x - y + 3 = 0$, which in the Euclidian geometry leads to the value of:

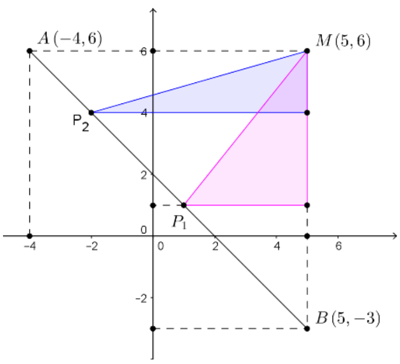
$$d = \frac{|2\cdot 1 - 1\cdot 1 + 3|}{\sqrt{4+1}} = \frac{4}{\sqrt{5}} \approx 1,78$$

In TG, the shortest way between $M(1,1)$ and the line $2x - y + 3 = 0$ we consider it to be MP, where *P* is an arbitrary point with the coordinates $P(\alpha, 2\alpha + 3)$.

$dt_{(P,M)} = |\alpha - 1| + |2\alpha + 3 - 1|$, with the solutions: $\begin{cases} -3\alpha - 1, \text{for } \alpha < -1 \\ \alpha + 3, \text{for } \alpha \in [-1,1] \\ 3\alpha + 1, \text{for } \alpha > 1 \end{cases}$

Representing these three variants in a system of coordinates, we observe that the minimum is obtained for $\alpha = -1$, so $P(-1,1)$ and the distance will be:

$$dt_{(P,M)} = |1+1| + |1-1| = 2$$



Using this numerical example, the distance between *M* and the given line in the Euclidian Geometry, is bigger than that in TG.

With another numerical example, we can establish that in TG, there is the possibility that the minimum distance between a point and a line can be any value from a finite interval. Let the given line be $x + y = 2$ and the point be $M(5,6)$. In the Euclidian Geometry:

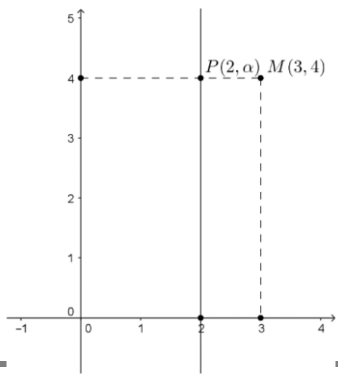
$$d = \frac{|5+6-2|}{\sqrt{2}} = \frac{9}{\sqrt{2}}$$

In TG, we calculate the distance *MP*, where $P(\alpha, 2-\alpha)$:

$d_{(P,M)} = |1-5| + |2 - \alpha - 6| = |\alpha - 5| + |\alpha + 4|$ with the solutions $\begin{cases} -2\alpha + 1, \text{for } \alpha \le -4 \\ 9, \text{ for } \alpha \in (-4,5) \\ 2\alpha - 1, \text{for } \alpha \ge 5 \end{cases}$

We observe that the minimum can be obtained for an infinite number of values $\forall \alpha \in [-4,5]$; we can also observe that the distance is, again, bigger than the one in the Euclidian Geometry.

Let's now consider a line parallel to Oy, the line $x = 2$ and the point $M(3,4)$. If we analyse this case, we can conclude that the



distance is equal, in both cases:
-In the Euclidian Geometry, MP is 1;
-In TG, we calculate the distance from M to $P(2,\alpha)$, using the formula:

$$dt_{(P,M)} = |3-2| + |4-\alpha| = 1 + |4-\alpha|$$

The minimum is obtained for $\alpha = 4 \Rightarrow MP = 1$

The parabola
A parabola consists a locus of points in a plan, determined by a point F (the focus) and line l(the directrix, F ϵ l) .

For a graphic representation in Taxicab Geometry of a parabola defined by the focus F(4,0) and the directrix x+4=0, we have to find the **locus of points** that is a fixed distance from F and l. In other words we are looking for M(x,y) so that d(M,F)=d(M,l).
Case 2

$y < 0 \Rightarrow |y| = -y$

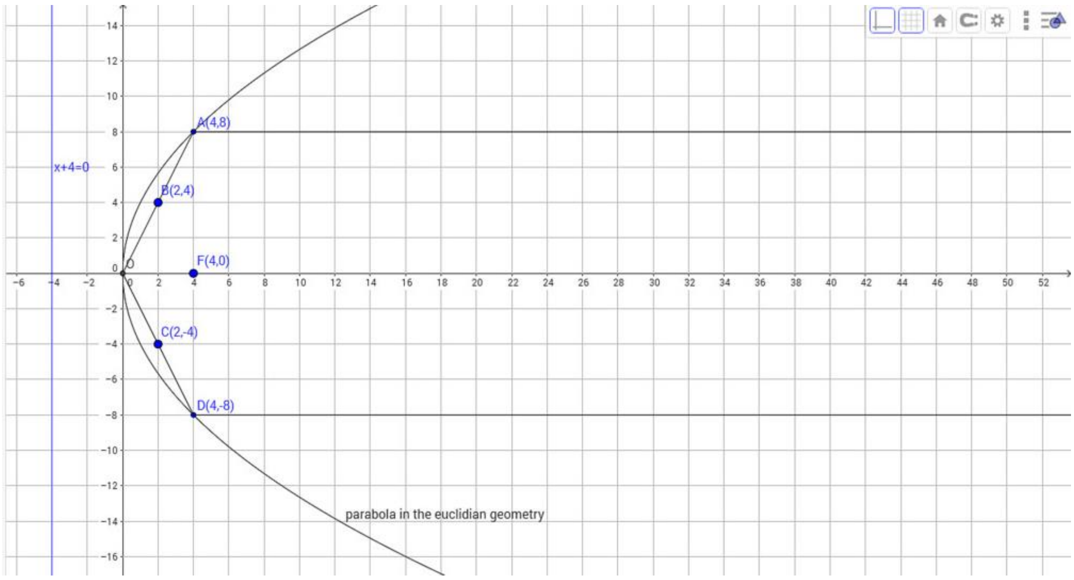
The solutions are similar with the ones above

$$-y = \begin{cases} 0, x \le 0 \\ 2x, x \in (0,4) \\ 8, x \ge 4 \end{cases} \Rightarrow y = \begin{cases} 0, x \le 0 \\ -2x, x \in (0,4) \\ -8, x \ge 4 \end{cases}$$

$$\left.\begin{array}{l} d(M,F) = |x| + |y-8| \\ d(M,l) = |x-8| \end{array}\right\} \Rightarrow |x| + |y-8| = |x-8|$$

Case 1
$y \ge 8 \Rightarrow |y-8| = y-8$ and now the equation is $y-8 = |x-8| - |x|$

$$\left.\begin{array}{ll} \text{a)} & x \le 0,\ y = -x+8+x+8 \\ \text{b)} & x \in (0,8), y = -x+8-x+8 \\ \text{c)} & x \ge 8,\ y = x-8-x+8 \end{array}\right\} \quad \begin{cases} y = 16,\ x \le 0 \\ y = -2x+16,\ x \in (0,8) \\ y = 0,\ x \ge 8 \end{cases}$$

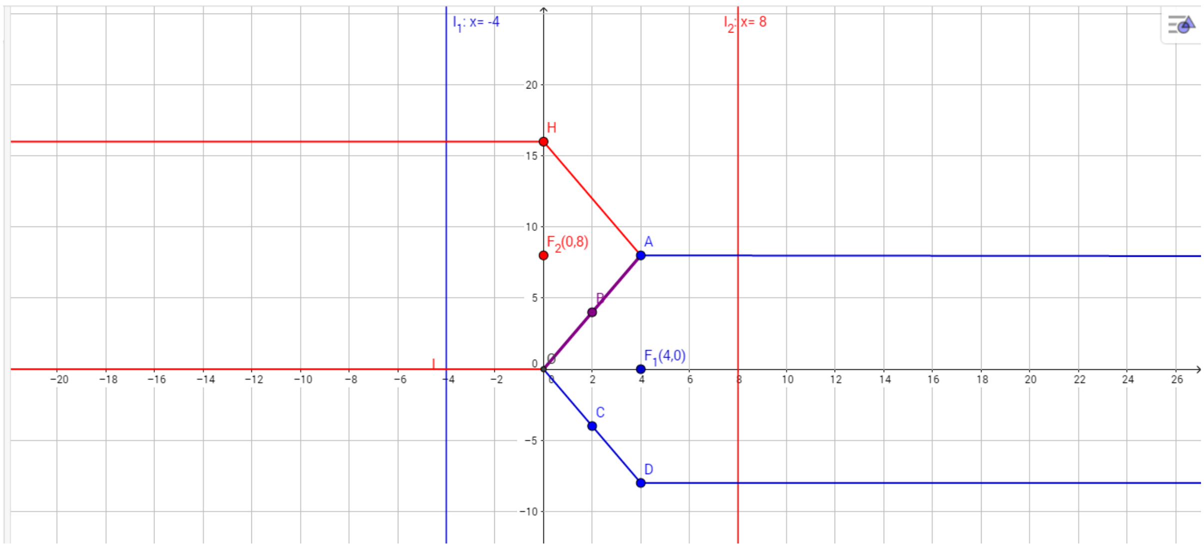


Case 2

$y < 8 \Rightarrow |y-8| = -y+8$

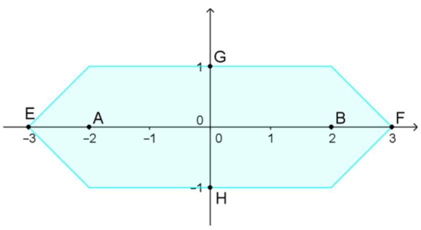
The solutions are similar with the ones above

$$-y=\begin{cases}16, x\leq 0\\ -2x+16,\ x\in(0,8)\\ 0, x\geq 8\end{cases} \Rightarrow y=\begin{cases}-16, x\leq 0\\ 2x-16, x\in(0,8)\\ 0, x\geq 8\end{cases}$$



8. Representing an ellipse in TG

The ellipse is defined in the Euclidian Geometry as being the “geometrical locus of all the points equally apart from two fixed points called focuses.” In order to represent an ellipse in TG with the focuses $A(-2,0)$ and $B(2,0)$ with the radius being 6, we must look for the points $M(X,Y)$ for which $d(M,A)+d(M,B)=6$, therefore:



For $y \geq 0 \Rightarrow |y| = y$; we have to represent the curve $y = \dfrac{6-|x+2|-|x-2|}{2}$, therefore:

$$y = \begin{cases} x+3 & x \in (-\infty, -2] \\ 1 & x \in (-2, 2) \\ 3-x & x \geq 2 \end{cases}$$

$$y \leq 0 \Rightarrow \begin{cases} x+3 \geq 0 \Rightarrow x \geq -3 \\ 3-x \geq 0 \Rightarrow x \leq 3 \end{cases}$$

$$\Rightarrow y = \begin{cases} x+3 & x \in [-3, -2] \\ 1 & x \in (-2, 2) \\ 3-x & x \in [2, 3) \end{cases}$$

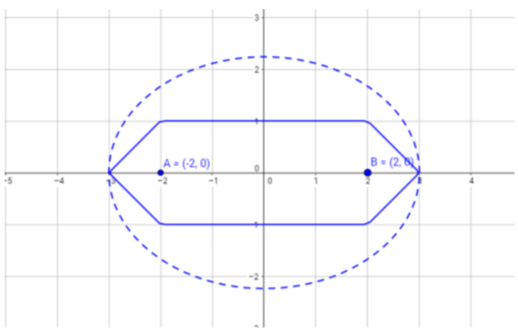
For $y < 0 \Rightarrow |y| = -y$, we explicit the module thusly:

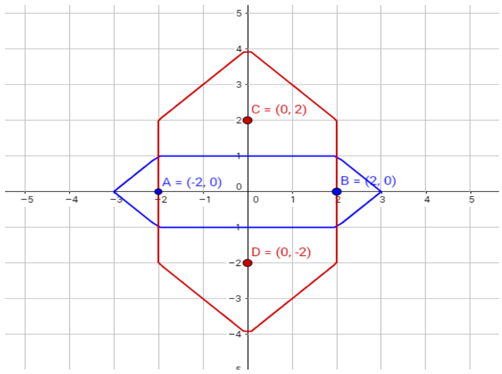
$$y = \begin{cases} -x-3 & x \leq -2 \\ -1 & x \in (-2, 2) \\ x-3 & x \in [2, \infty) \end{cases}$$

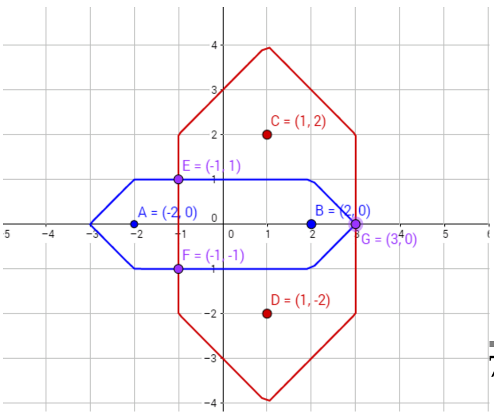
$$y \leq 0 \Rightarrow \begin{cases} -x-3 \leq 0 \Rightarrow x \geq -3 \\ x-3 \leq 0 \Rightarrow x \leq 3 \end{cases}$$

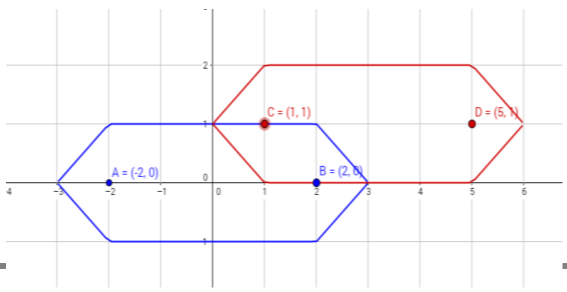
$$\Rightarrow y = \begin{cases} -x-3 & x \in [-3, -2] \\ 1 & x \in (-2, 2) \\ 3-x & x \in [2, 3) \end{cases}$$

Note: The ellipse's axis of symmetry are Ox and Oy.
From these considerations, we can study the 2D mode of representation of other curves (the hyperbole, the parabola, the Fibonacci spiral, trigonometrical functions, etc.) and geometrical figures.









Conclusion

Although, seemingly without any sort of profundity to it, the Taxi Geometry offers a new perspective over this particular section of mathematics. From changing famous numerical constants and the appearance of shapes, to finding different solutions to problems of practical and impractical use, this subject is endless and creates an infinite number of new dilemmas for mathematicians to solve. Born in Manhattan, due to the necessity of finding the exact distance between two points (given that this was before the era of Google Maps), this new way of thinking about distances is able to determine a radical switch in the manner in which we regard geometry as a whole. For millennia, we have calculated the distance between 2 points as being the exact straight line between the two, in essence the hypotenuse of a right-angle triangle. Why shouldn’t it be the sum of the catheti’s lengths?

The Bestiary of Maria-José

2016- 2017

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Abstract

In this document we have tried to solve the first one of the two problems submitted by the Math.En.Jeans Association: the bestiary of Marie-José.

The first problem deals with the theory of the tree graph, in particular with the number of nodes and the height of the same graph. We will conclude that the first problem is similar to the study of Dyck paths or to the disposition of nested parenthesis.

We'd like to thank professor Giacon of Liceo Nievo and all his students for their kind support.

We'd like to thank professor Zanardo of the Dipartimento di Matematica "Tullio Levi Civita" dell'Università degli Studi di Padova for his support and help.

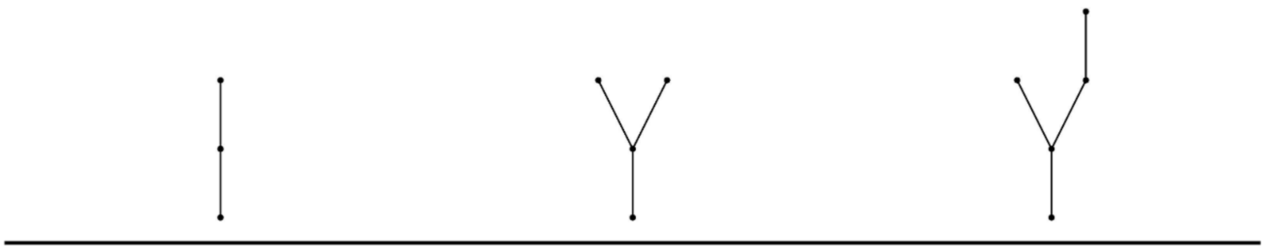
1 The first problem: the bestiary of Marie-José

The hydra is a very rare mythical animal. We do not know much about this living being. We know that, when they are born, hydras have only one head, anchored in the ground.

Then they grow up in two different ways:

- a node is formed on the base of a head, and a neck stretches between the head and the node;
- a new head appears on an existing node, and a neck stretches between the new head and the node.

In the following example, we show different ways in which a hydra can grow:



Since last June, Marie-José has started collecting hydras. She wants to show them on the walls, so she decided to let them grow up into frames to have some nice pictures. In this way, for a certain hydra, different possible pictures exist.

Marie-José already owns many different hydras. To classify them, she decided to measure the growth of her hydras on the base of their size, considering both the heads and the nodes. The size n of a hydra is the sum of the number of the heads and the nodes.

Marie-José, who also likes maths, wants to know if there is a formula, that depends on the number n, that provides the number of different pictures that contain a size n hydra. For example, there is only one picture of a size 1 hydra, and only one picture of a size 2 hydra. Conversely, there are two possible pictures of a size 3 hydra.

Hercules, after a personal bad experience, prefers hydras with maximum height 2 (the height of a neck is 1). Does a formula that allows to determine the number of pictures that contain the size n and height 2 hydra exist? The same question for the size n and height k hydras.

2 Discussion

First of all, we notice that a hydra could be represented with a tree graph, that is a planar graph that divides the plan in only one surface.

According to the Euler relation:

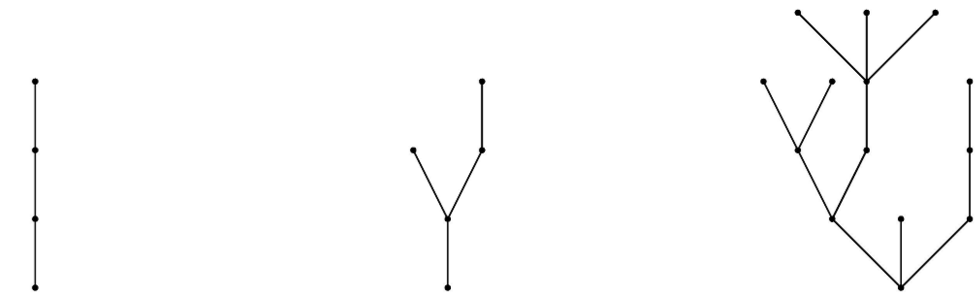
$$f + n - s = 2$$

where: f is the number of faces[1], n is the size (number of heads and necks), s is the number of necks. In our case, we note that $s = n - 1$.

Each graph has a base node (the only head the hydra owns at first). For each other node we define as *height* the number of necks between the base node and itself. We call height k of a hydra the maximum value of the heights of single nodes.

Example:

$n = 4$ $k = 3$	$n = 5$ $k = 3$	$n = 14$ $k = 4$



3 Hydra Classes

It is useful, to calculate the number of possible hydras, to divide them into classes.

We call *class* a defined sequence of natural, not null numbers $\{x_1, \ldots, x_k\}$ where the element x_i identifies the total number of branches (necks) at height i, regardless of their location.

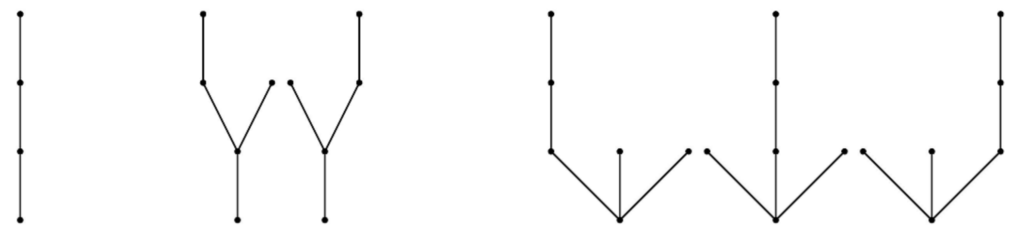
So, we have:

$$s = n - 1 = x_1 + \cdots + x_k$$

We observe that, in no trivial cases, the classes are formed by several distinct hydras.

Examples:

Class $\{1,1,1\}$	Class $\{1,2,1\}$	Class $\{3,1,1\}$

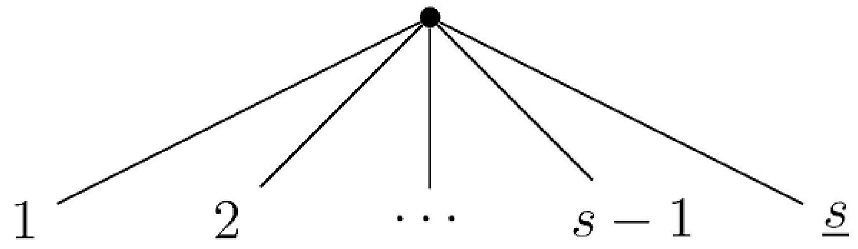


[1] In our case, trivially, $f = 1$.

Tree that generates classes

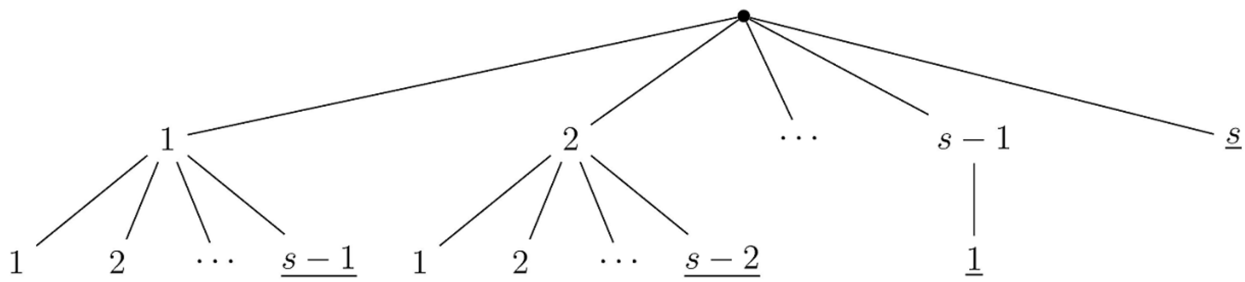
To create all the possible hydra classes with s and k assigned, the procedure is as follows:

On the first level, we can place from 1 to s branches:



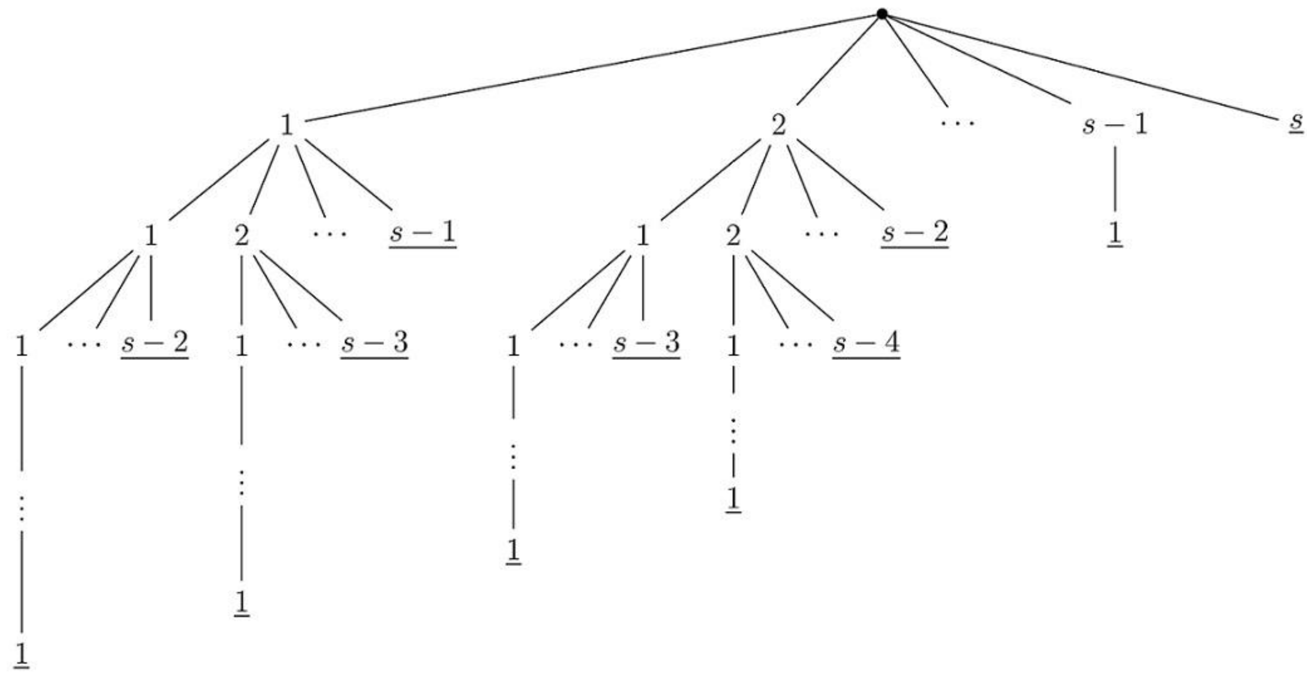
If we place exactly s branches, we won't have any more branches available for the next levels. We note that we have already allocated the desired number of branches. We obtain only one class $\{s\}$.

Instead in the remaining $s-1$ cases at least one branch is still available to be placed on the next level. Therefore, we can continue by placing, wherever possible, further branches on the second level.



On the second level $s-1$ new classes have been formed. Their type is $\{i, s-i\}$ with $1 \leq i \leq s-1$.

We go on recursively to the construction of the tree. The final tree will be as follow:

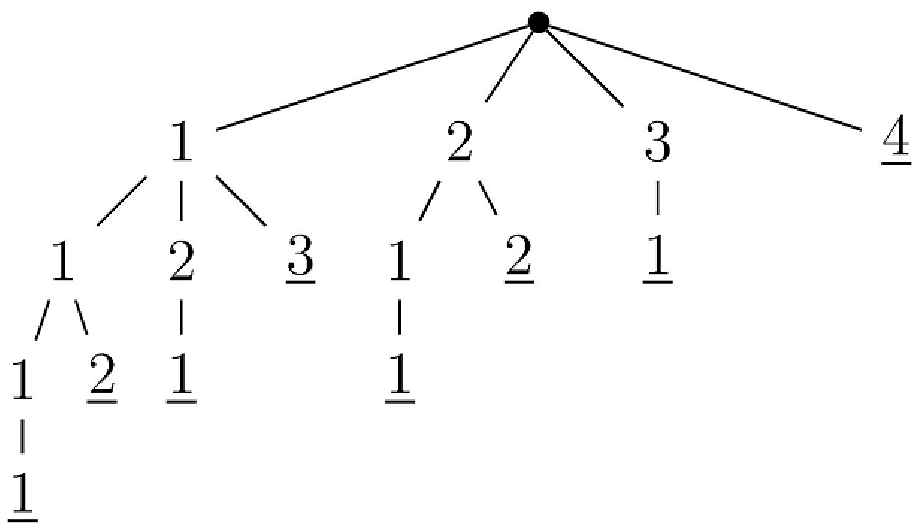


Note that the values for which the sum of the branches for a given possibility is s and for which the class is complete (values that coincide with terminal nodes of the graph) have been underlined.

To enumerate every class, it is enough to read the graph of classes from top to bottom in every possible way that conduct to an underlined number. If we want to consider a given height class only, or height less than or equal to the given k, we only take the classes that end respectively at the k-th point, or in the first k points.

Example

For $s = 4$ and $k \leq 3$ the tree is as follows:



And so, the classes will be:

$$\{1,1,2\},\{1,2,1\},\{1,3\},\{2,1,1\},\{2,2\},\{3,1\},\{4\}$$

4 Possibility for a given class

There is now the problem of determining how many different graphs it is possible to build for a given class $\{x_1,\dots,x_k\}$. The procedure is as follows.

For the first level, independently from x_1 value, we have just one possibility of drawing $x_1$branches.

For the i-th level we have more possibilities of placing the x_i branches in the x_{i-1} nodes that the branches of the lower level make available. So the problem is to calculate the ways of varying x_{i-1} integer numbers (the number of branches at the i-th level for every node of the previous level) positive or null, so that their sum is x_i (the number of branches we have to place at the i-th level).

In order to do this, we consider $x_i+x_{i-1}-1$ white squares and we blacken $x_{i-1}-1$ of them.

$$\overbrace{\square\blacksquare\square\square\blacksquare\square\square\cdots\square\square\square\blacksquare}^{x_i+x_{i-1}-1\text{ squares}}$$

We observe that anyway we blacken the squares, x_i of them remain. These ones represent the sum of the x_{i-1} numbers, that we want in fact to be x_i. The black squares are dividers. This means they divide the sequence of the squares into $(x_{i-1}-1)+1=x_{i-1}$ areas (eventually null) that represent the single numbers whose sum has to be x_i. Each of these numbers represent the number of branches we are going to draw in order at every node of the previous level.

The possibilities of placing the branches are the same as the possibilities of choosing $x_{i-1}-1$ elements (the black squares) from a set that contains $x_i+x_{i-1}-1$, that is $\binom{x_i+x_{i-1}-1}{x_{i-1}-1}$.

The last thing to do is to do the calculation for every level of the class and multiply the results.

$$p\{x_1,\cdots,x_k\}=\prod_{i=2}^{k}\binom{x_i+x_{i-1}-1}{x_{i-1}-1}$$

Example

For $s=4$ and $k\leq 3$ we have the following classes:

$$\{1,1,2\},\{1,2,1\},\{1,3\},\{2,1,1\},\{2,2\},\{3,1\},\{4\}$$

$$\begin{aligned}
p\{1,1,2\} &= \binom{1+1-1}{1-1}\binom{2+1-1}{1-1} = 1\cdot 1 && = 1\\
p\{1,2,1\} &= \binom{2+1-1}{1-1}\binom{1+2-1}{2-1} = 1\cdot 2 && = 2\\
p\{1,3\} &= \binom{3+1-1}{1-1} && = 1\\
p\{2,1,1\} &= \binom{1+2-1}{2-1}\binom{1+1-1}{1-1} = 2\cdot 1 && = 2\\
p\{2,2\} &= \binom{2+2-1}{2-1} && = 3\\
p\{3,1\} &= \binom{1+3-1}{3-1} && = 3\\
p\{4\} &= && = 1
\end{aligned}$$

So, the possibilities will be 1+2+1+2+3+3+1=13.

5 General Formula

Finally, to calculate the possible given graphs for given s and k, we find all the possible classes at first (in case choosing by height), then we calculate the possibilities for each one with the previous formula and we sum all the results.

We can therefore write that the number $N(n,k)$ of hydras of size n and height k is

$$N(n,k) = \sum_{\substack{x_1+\cdots+x_k=n-1 \\ 1\leq x_1,\ldots,x_k\leq n-k}} p\{x_1,\ldots,x_k\}$$

To realize this computation, we use an algorithmic procedure that allows to permute the values $x_1,\ldots,x_k$ in every possible way, to calculate the possibilities for each class $p\{x_1,\ldots,x_k\}$ and to sum the obtained results.

The scheme of the algorithm in pseudocode is the following:

```
poss := 0

function permute : (x_1, ··· , x_k), length , partialSum

   if length = 1
      x_k := partialSum
      poss := poss + p{x_1, ··· , x_k}
      return
   end

   for i ∈ ℕ, 1 ≤ i ≤ partialSum - 1
      x_{k−length} := i
      permute : (x_1, ··· , x_k), length - 1, partialSum - i
   end

end
```

The permute function fixes the first element of the analyzed class and, recursively, makes the values of the sub-class obtained cycle by excluding the first element (a similar problem to the starting one). That sub-class has, as sum s of the elements, the sum of the elements of the starting class whence the excluded element has been subtracted. This procedure continues until we analyze a class with a single element. In that case, the class value will be the only possible value (partialSum) that is now equal to $s-(x_1+\cdots+x_{k-1})$, where $s=k-1$ is the sum of the elements of the starting class.

The starting call, fixed n and k, will be:

permute: (0, … , 0), k , n-1

In the end, we will have:

poss = N(n, k)

6 Output

Here are the output values of the algorithm for $n \leq 25$ and $k \leq 6$:

n	$N(n,2)$	$N(n,3)$	$N(n,4)$	$N(n,5)$	$N(n,6)$
3	1				
4	3	1			
5	7	5	1		
6	15	18	7	1	
7	31	57	33	9	1
8	63	169	132	52	11
9	127	482	484	247	75
10	255	1341	1684	1053	410
11	511	3669	5661	4199	1975
12	1023	9922	18579	16017	8778
13	2047	26609	59917	59224	36938
14	4095	70929	190696	214058	149501
15	8191	188226	600744	760487	587951
16	16383	497845	1877256	2665884	2262375
17	32767	1313501	5828185	9246276	8558854
18	65535	3459042	17998783	31793724	31945379
19	131071	9096393	55342617	108548332	117939506
20	262143	23895673	169552428	368400045	431530926
21	524287	62721698	517884748	1244027317	1567159901
22	1048575	164531565	1577812060	4182854728	5655480303
23	2097151	431397285	4796682165	14012220027	20299352107
24	4194303	1130708866	14555626635	46789129817	72522832282
25	8388607	2962826465	44100374341	155798575851	258054207727

7 Case k = 2

In the particular case of $k = 2$ it is possible to find an explicit formula for the number of hydras.

We already know from section 3 that all the height 2 classes are of the type $\{x_1, n-1-x_1\}$, with $1 \leq x_1 \leq s-1$. In addition, $\sum_{\kappa=0}^{\nu}\binom{\nu}{\kappa} = 2^{\nu}$. So, we have:

$$\begin{aligned} N(n,2) &= \sum_{\substack{x_1+x_2=n-1 \\ 1\leq x_1,x_2\leq n-2}} p\{x_1,x_2\} \\ &= \sum_{x_1=1}^{n-2} p\{x_1, n-1-x_1\} \\ &= \sum_{x_1=1}^{n-2} \binom{n-2}{x_1-1} \\ &= \sum_{x_1=1}^{n-1} \binom{n-2}{x_1-1} - \binom{n-2}{n-2} \\ &= 2^{n-2}-1 \end{aligned}$$

8 Isomorphisms

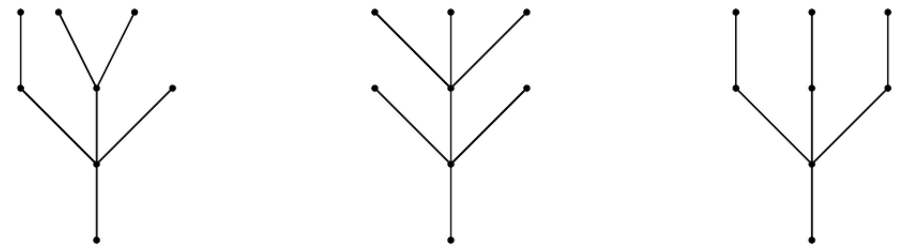
8.1 Nested Parentheses

It is possible to associate uniquely a hydra to a series of balanced parentheses (for every open parenthesis there is a closed one) and nested.

- Each depth level corresponds to a height;
- Adjacent couples of parentheses, at the same depth, represent necks that start from the same node.

Examples:

$$\Big((())(()())()\Big) \qquad \Big(()(()()())()\Big) \qquad \Big((())(())(())\Big)$$



We know[2] that the number of ways we can place s balanced parentheses is the number of Catalan C_s,defined as:

$$C_s = \frac{1}{s+1}\binom{2s}{s}$$

We indicate with $N(n)$ the number of possible hydras of class n, with a heigth between 1 and $n-1$, that is:

$$N(n) = \sum_{k=0}^{n-1} N(n,k)$$

Hydras correspond to nested parenthesis, so:

$$N(n) = C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1} = \frac{(2n-2)!}{n\big((n-1)!\big)^2}$$

A confirmation of the last formula can be obtained by doing the calculation of $N(n)$ manually; it is the same as summing by rows the values of the table in section 6.

The result is the following list, that is the start of the sequence of Catalan numbers:

n	$N(n)$	n	$N(n)$
2	1	14	742900
3	2	15	2674440
4	5	16	9694845
5	14	17	35357670
6	42	18	129644790
7	132	19	477638700
8	429	20	1767263190
9	1430	21	6564120420
10	4862	22	24466267020
11	16796	23	91482563640
12	58786	24	343059613650
13	208012	25	1289904147324

8.2 Dyck's paths

Another way we can imagine the problem involves Dyck's paths. At first, we define Dyck's paths as the "steps" over the diagonals of the square of a grid $(x,y) \in \mathbb{N}^2$ such that:

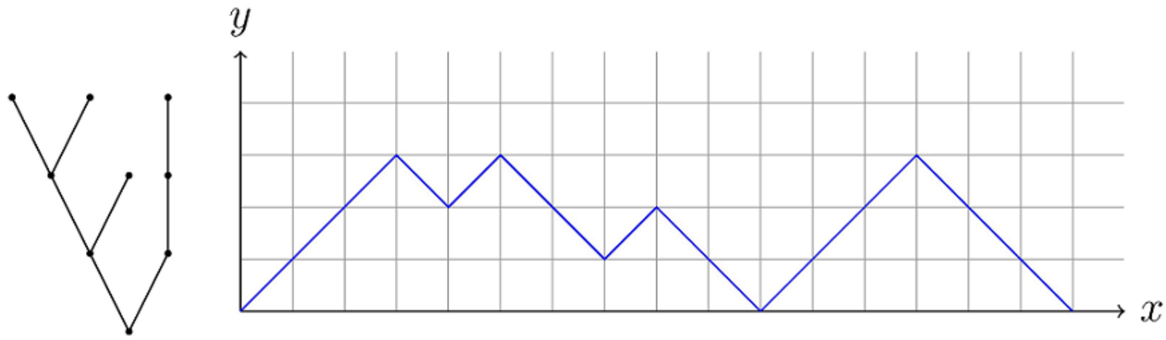
- The starting point is $(0,0)$ and the end is $(2s,0)$ with $s = n-1$;
- If a point (x_0, y_0) is part of the path, the following point can only be (x_0+1, y_0+1) or (x_0+1, y_0-1).

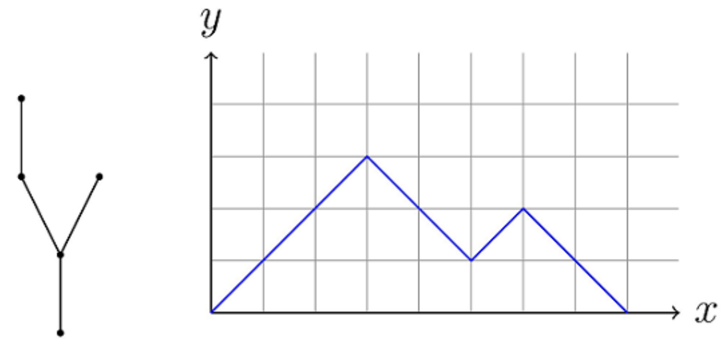
To draw a hydra as a path we use these rules:

[2] See: https://en.wikipedia.org/wiki/Catalan_number\#Applications_in_combinatorics

- We go through every branch of the hydra, starting from the base, bottom to top and left to right. Having reached a head, we come back to the first node with any ramification and we continue the same way with each of them. In the end, we will return to the base node. In this way, every neck of the hydra will have been covered twice (that is why the path is $2s$ steps long);
- Each movement on the hydra (upward or downward) is represented by a step to the right in the path, (upward or downward).

Examples:





As before, we know[3] that the number of possible Dyck's paths fixed the length $2s$ is exactly the Catalan number C_s.

[3] Weisstein, Eric W. ``Dyck Path." From MathWorld -- A Wolfram Web Resource. http://mathworld.wolfram.com/DyckPath.html

This article is written by students. It may include omissions and imperfections.

The Fastest Path

2016- 2017

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The fastest path between two points in different situations

In our project, we wanted to solve two applied mathematics problems concerning the shortest path in time between two points in different situations. Hereby we are going to show their setting as well as their solutions.

1. The lifeguard problem

We consider that a lifeguard on the beach sees a victim on the edge of drowning in a calm water (lake/sea). We know the distance between the position of the lifeguard and the shore, his speed on the beach, and the distance between the point where the victim is and the shore, as well as the lifeguard’s speed in the water. The time loss while the lifeguard enters the water is considered irrelevant. He must choose the fastest path to the victim.

In this situation, A is the lifeguard’s position, E-its projection on the shore, B-the victim’s position, and F-its projection on the shore. AE=a, BF=c, EF=b;

v1= speed on the beach.

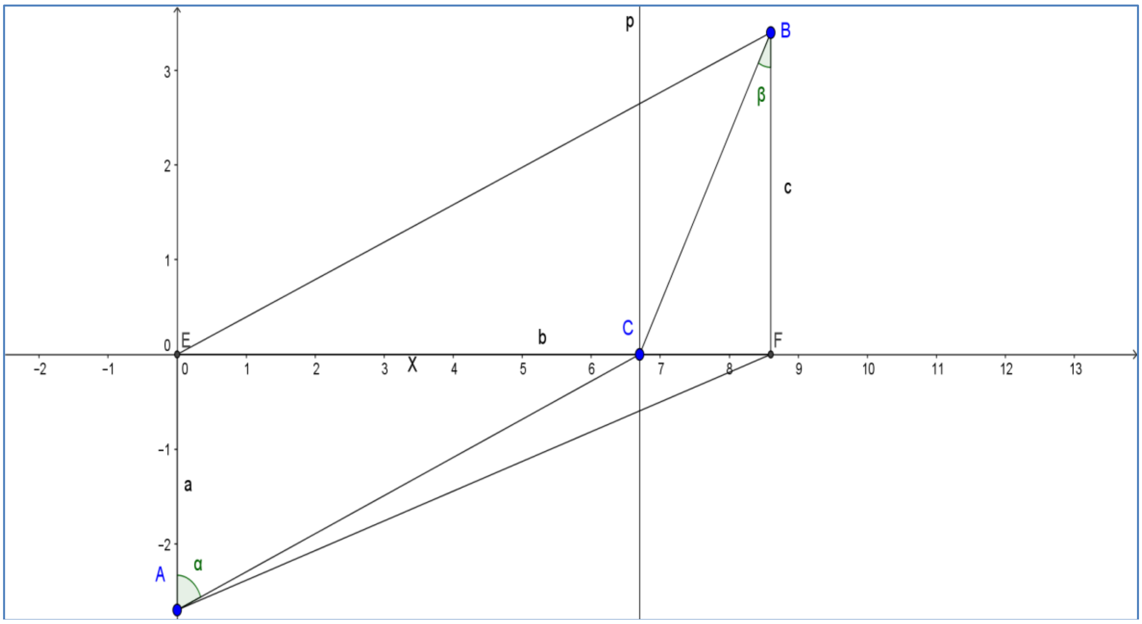
v2= speed in the water.

Our trials

The complete duration of the lifeguard’s movement is:

$$\frac{AC}{v1}+\frac{BC}{v2}=\frac{\sqrt{x^2+a^2}}{v1}+\frac{\sqrt{(b-x)^2+c^2}}{v2}$$

After days of trials, we were unable to draw any conclusion. Then, we found out that the derivatives are essential in the solving of this problem.



So the total time can be calculated using the formula:

$$f(x)=\frac{AC}{v1}+\frac{BC}{v2}=\frac{\sqrt{x^2+a^2}}{v1}+\frac{\sqrt{(b-x)^2+c^2}}{v2};$$

Now, let's calculate the f derivative:

$$f'(x)=\left(\frac{\sqrt{x^2+a^2}}{v1}\right)'+\left(\frac{\sqrt{(b-x)^2+c^2}}{v2}\right)'=\left(\frac{1}{v1}\sqrt{x^2+a^2}\right)'+\left(\frac{1}{v2}\sqrt{(b-x)^2+c^2}\right)'=$$

$$=\frac{1}{v1}\cdot\frac{1}{2\cdot\sqrt{x^2+a^2}}\cdot 2x-\frac{1}{v2}\cdot\frac{1}{2\sqrt{(b-x)^2+c^2}}\cdot 2(b-x)$$

The time is at its lowest when f'(x) =0, that is:

$$\frac{1}{v1}\cdot\frac{1}{2\cdot\sqrt{x^2+a^2}}\cdot 2x=\frac{1}{v2}\cdot\frac{1}{2\sqrt{(b-x)^2+c^2}}\cdot 2(b-x) <=>$$

$$\frac{x}{v1\sqrt{x^2+a^2}}=\frac{b-x}{v2\sqrt{(b-x)^2+c^2}} <=>$$

$$\frac{EC}{v1\cdot AC}=\frac{CF}{v2\cdot CB} <=> \frac{v1}{v2}=\frac{\sin\alpha}{\sin\beta}$$

<u>Interesting similarities</u>

The relation obtained after solving the lifeguard problem bears a striking resemblance to the refraction law in optics. In fact, the relations are equivalent because the light rays always obey this law, following the "shortest path in time", just as our lifeguard (in accordance with Fermat's principle).

<u>2. The circle chords</u>

We consider a vertical plane containing a circle, and all its chords can be taken as inclined planes. A ball slides on a chord from the circle's top, without friction. What is the relation between the times for different chords?

First of all, we consider g-the gravitational acceleration. We also consider A as the top of the circle, and C and D the two arbitrary points on the circle given in the problem.

AC=l; AO=OB=R=AB/2; AD=i;

The descending acceleration for a plane with the angle k, is g*sin k

As the movement has no friction, the time, t1, from A to C, can be calculated with the formula:

$$l=\frac{1}{2}*g*sin\alpha 1*t1^2$$

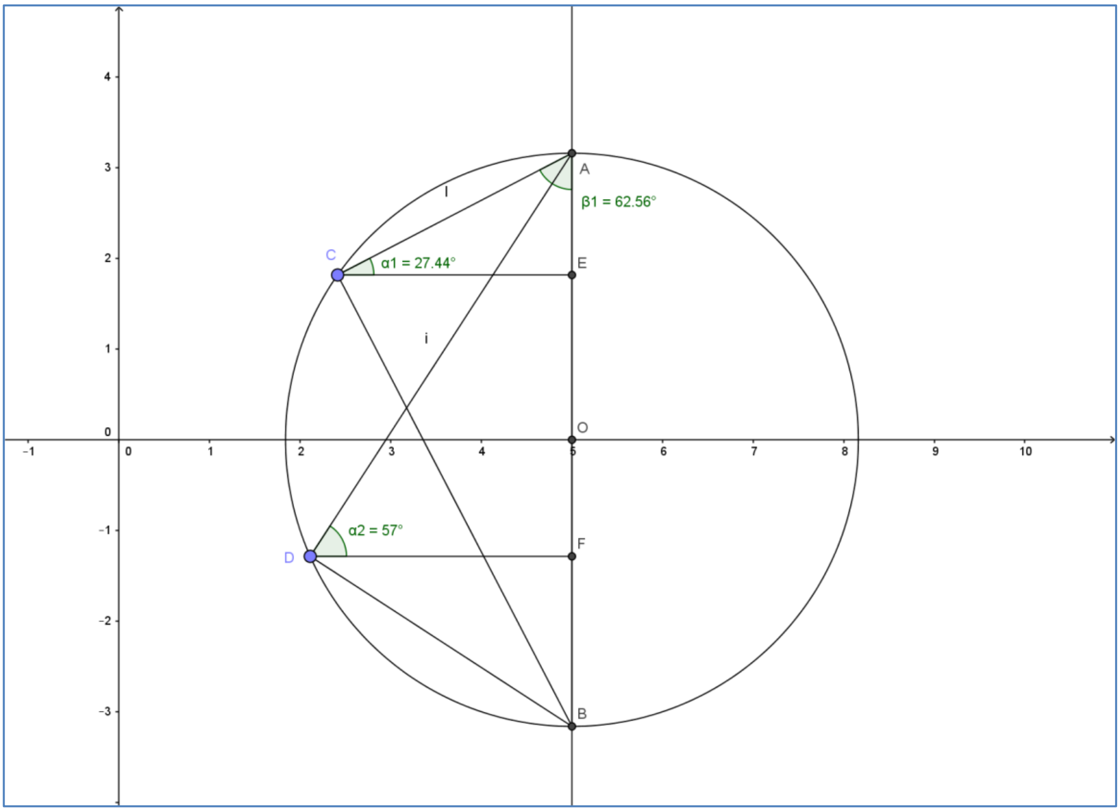
But as both angles, ACB and ADB, have the measurement equal to 90°, because they determine a circle arch of 180°, then $sin\alpha 1 = cos\beta 1 = \frac{l}{2R}$. Introducing this in the equation above, we get to:

$$t1 = 2\sqrt{\frac{R}{g}}$$

We can observe that the result depends neither on $l1$, nor on $\alpha 1$.
Similarly, we can determine the time from A to D,

$$t2 = 2\sqrt{\frac{R}{g}}$$

So, $t1 = t2$, the time periods needed to reach C and D from A are equal.
The time needed for descending on any random chord does not differ from chord to chord.



In conclusion to the first problem, we can say that the shortest path is found when the ratio of the sinuses of the angles is equal to the ratio of the correspondent speeds on the two surfaces.
In conclusion to the second problem, we can state that whichever the chord from the top point to one on the circle is, the descending times do not differ.

Future plans
We would like to study further ideas for this kind of problems, for example the fastest path between two given points on a circle situated in a vertical plane.

This article is written by students. It may include omissions and imperfections.

The Game of Differences

2016-2017

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Research topic:

There are 4 natural numbers a, b, c, d written on a row. The differences |a-b|, |b-c|, |c-d|, |d-a| are then written on the next row. The process is continued:

7	3	9	2
4	6	7	5
2	1	2	1
1	1	1	1
0	0	0	0

It can be observed that the null row 0, 0, 0, 0 has been obtained.

- Is this a coincidence or is the null row obtained for any natural numbers a, b, c, d?
- What happens if the rows don't have 4 numbers, but 3, 5, 6, …?
- What happens if we have 4 rational numbers
- What happens if we have 4 real numbers

Brief presentation:

- We solved the natural numbers' case using remainders modulo 2.
- We also treated the cases of 3, 4, 5, 6, 7, 8, … natural numbers on the first row, obtaining different conclusions, afterwards we made a C++ program.

- We extended the case of integers, followed by rational numbers.
- Subsequently, we prove significant results for real numbers.

Solution:

What happens if we have:

1. 4 natural numbers
2. 5, 6, 7, ... natural numbers
3. 4 rational numbers
4. 4 real numbers

1.

1.1 Solution 1:

We replace the numbers with their remainders modulo 2. There are 2^4=16 ways to distribute the remainders 0 and 1 on the first row. All these possibilities are consisted in the following table:

1	0	1	1	1	1	0	1	1	1	1	0	0	1	1	1
or				or				or				or			
0	1	0	0	0	0	1	0	0	0	0	1	1	0	0	0
1	1	0	0	0	1	1	0	0	0	1	1	1	0	0	1
0	1	0	1	1	0	1	0	0	1	0	1	1	0	1	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

So, after at most 4 steps, the obtained numbers will be divisible by 2. The remainders at the division by 2^2=4 when we divide the last numbers obtained are 0 or 2. If we make similar tables with the one made before, but instead of using the pair (0,1) we used the pair (0,2) of remainders, we notice that, after at most another 4 steps, all the numbers obtained will be divisible by 4.
We continue the process. **After at most 4k steps, all the numbers on the last row will be divisible by 2^k.**
So, the numbers on the last row will be divisible with an arbitrarily big power of 2. In conclusion, after a certain number of steps, we will reach the zero row: 0, 0, 0, 0.

1.2 Solution 2:

Let $a_1 = a, b_1 = b, c_1 = c, d_1 = d$ and $a_n, b_n, c_n, d_n \in \mathbb{N}$, $a_n = |a_{n-1} - b_{n-1}|$, $b_n = |b_{n-1} - c_{n-1}|$, $c_n = |c_{n-1} - d_{n-1}|$, $d_n = |d_{n-1} - a_{n-1}|$.

Let $x_n = \max\{a_n, b_n, c_n, d_n\}$. Taking into consideration the way sequences are defined and the fact that we are working with positive integers, it can be demonstrated through mathematical induction that the sequence x_n is decreasing. To demonstrate that, on the last row, at some point, there will only be values of zero, we should prove that the sequence cannot stagnate for positive values.

Let $a_{n+1} = x_n$. We will demonstrate that the maximum cannot stagnate for more than 4 steps for positive values.

If $a_{n+5} < a_{n+1}$, the conclusion is true.

If $a_{n+5} = a_{n+1}$, as $a_{n+5} = |a_{n+4} - b_{n+4}|$, with $0 \le a_{n+4} \le a_{n+1}$ and $0 \le b_{n+4} \le a_{n+1}$ (natural numbers), it necessarily ensues that $a_{n+4} = a_{n+1}$ and $b_{n+4} = 0$, or $a_{n+4} = 0$.

I. If $a_{n+4} = a_{n+1}$ and $b_{n+4} = 0$:

But how $b_{n+4} = |b_{n+3} - c_{n+3}|$ and $a_{n+4} = |a_{n+3} - b_{n+3}|$, it results that $b_{n+3} = c_{n+3}$ and $|a_{n+3} - b_{n+3}| = a_{n+1}$, and as $a_{n+3}, b_{n+3} \in \mathbb{N}$, $a_{n+3}, b_{n+3} \le a_{n+1}$, we can deduce two subcases:

1. $a_{n+3} = a_{n+1}, b_{n+3} = 0$ or
2. $a_{n+3} = 0, b_{n+3} = a_{n+1}$

We study each subcase:

1. $a_{n+3} = a_{n+1} \Rightarrow |a_{n+2} - b_{n+2}| = a_{n+1}$, and $b_{n+3} = 0 \Rightarrow b_{n+2} = c_{n+2}$ and it results $c_{n+3} = 0 \Rightarrow c_{n+2} = d_{n+2}$.

   $|a_{n+2} - b_{n+2}| = a_{n+1}$ ensues:

   i. $a_{n+2} = a_{n+1}$ and $b_{n+2} = c_{n+2} = d_{n+2} = 0$, which means that all numbers will become 0.

   ii. $a_{n+1} = b_{n+1}$ and $b_{n+2} = c_{n+2} = d_{n+2} = a_{n+1}$, which means that all numbers will become 0.

2. $a_{n+3} = 0 \Rightarrow a_{n+2} = d_{n+2}$ and $b_{n+3} = a_{n+1} \Rightarrow |b_{n+2} - c_{n+2}| = a_{n+1}$ and $c_{n+3} = a_{n+1} \Rightarrow |c_{n+2} - d_{n+2}| = a_{n+1}$.

   i. $b_{n+2} = a_{n+1}, c_{n+2} = 0$ and $d_{n+2} = a_{n+2} = a_{n+1}$, which leads to 0.

   ii. $c_{n+2} = a_{n+1}, b_{n+2} = 0$ and $c_{n+2} = a_{n+1}, d_{n+2} = 0$, which leads to 0.

If $b_{n+4} = a_{n+1}, a_{n+4} = 0$. Analogous.

2. What happens if on the first line there are not four numbers, but three, five, six or seven?

We will show you that it is possible that, for certain configurations of the first line, to not achieve the zero configuration at the end.

But, as said before, what is going to happen if we choose to have a different amount of numbers on the first line?

We are going to validate that for some ways of choosing the configurations on the first line, we will enter in a cycle. So, it’s obvious that no matter what we won’t achieve the zeroes.

If there are 3 numbers on a row:

0	1	1
1	0	1
1	1	0
0	1	1

- cycle of length 3 -

If there are 5 numbers on a row:

0	0	0	1	1
0	0	1	0	1
0	1	1	1	1
or				
1	0	0	0	0
1	0	0	0	1
		or		
0	1	1	1	0
1	0	0	1	0
		or		
0	1	1	0	1
1	0	1	1	1
		or		
0	1	0	0	0
1	1	0	0	0
		or		
0	0	1	1	1
0	1	0	0	1
		or		
1	0	1	1	0
0	1	1	0	0
		or		
1	0	0	1	1
1	0	1	0	0
		or		
0	1	0	1	1
1	1	1	0	1
		or		
0	0	0	1	0

or				
0	0	1	1	0
0	1	0	1	0
or				
1	0	1	0	1
1	1	1	1	0
0	0	0	1	1

- cycle of length14 -

If there are 6 numbers on a row:

0	0	0	1	0	1
0	0	1	1	1	1
0	1	0	0	0	1
1	1	0	0	1	1
0	1	0	1	0	0
1	1	1	1	0	0
0	0	0	0	0	1

- cycle of length 6 -

If there are 7 numbers on a row:

0	0	0	0	0	1	1
0	0	0	0	1	0	1
0	0	0	1	1	1	1
0	0	1	0	0	0	1
0	1	1	0	0	1	1
1	0	1	0	1	0	1
1	1	1	1	1	1	0
0	0	0	0	0	1	1

- cycle of length 7 -

If on the first line there are eight numbers, they will become 0 after a number of steps as well.

To prove this, we have written a C++ program that demonstrates that, no matter the order of the zeros and ones on the first line of the table, they will become zeros in the end. Using the idea from the first solution, we replace the numbers with their remainder after the division by 2. In maximum of 8 steps they will become zero, so the numbers will be divisible by 2. Continuing the reasoning, it results that the numbers will become 0 after a maximum of 8k steps. The program generates a first the numbers between 0 and 255 and converts them to binary to obtain all the 8 digit numbers that represent the possible combinations of 0 and 1. Then, it implements the algorithm described in the rubric, until it reaches a null row or until it becomes a cycle. To be able to check at the end if all the combinations check the condition, we have taken a variable ‘ok’, which increases each time a combination checks the condition. Result: The condition is true for all combinations.

3.What happens if on the first line there are four rational numbers?

The set of elements M = (a, b, c, d) and tM = (ta, tb, tc, td) have the same lifespan. If we start from four rational numbers, we multiply them by the lowest common multiple of denominators and we get a first row of four integers, which, after a certain number of steps will consist of just zeros.

Let us consider $X=b_1\times b_2\times b_3\times b_4$, $n\neq 0$ and $b_1,b_2,b_3,b_4\neq 0$.

$\frac{a_1}{b_1}$	$\frac{a_2}{b_2}$	$\frac{a_3}{b_3}$	$\frac{a_4}{b_4}$
d_1	d_2	d_3	d_4
.	.	.	.
.	.	.	.
.	.	.	.
.	.	.	.
.	.	.	.
x_1	x_2	x_3	x_4

We multiply the numbers from the first column and we get the following table:

$a_1\times b_2\times c_3\times d_4 \in$ N or Z
$a_2\times b_1\times b_3\times b_4 \in$ N or Z
$a_3\times b_1\times b_2\times b_4 \in$ N or Z
$a_4\times b_1\times b_2\times b_3 \in$ N or Z

$a_1\times b_2\times b_3\times b_4$	$a_2\times b_1\times b_3\times b_4$	$a_3\times b_1\times b_2\times b4$	$a_4\times b_1\times b_2\times b_3$
$n\times d_1$	$n\times d_2$	$n\times d_3$	$n\times d_4$
.	.	.	.
.	.	.	.
.	.	.	.
.	.	.	.
.	.	.	.
$n\times x_1$	$n\times x_2$	$n\times x_3$	$n\times x_4$

Knowing now that we start from 4 integers, following the remainder judgment of the division by 2, we will obtain four 0.
$\rightarrow n\times x_1 = n\times x_2 = n\times x_3 = n\times x_4 = 0$
$n\neq 0$
$\rightarrow x_1= x_2= x_3= x_4 = 0 \rightarrow$ and in the first table, after a certain number of steps, we will obtain just 0.

4. Is our result true if we choose some random four real numbers for the first line?

The answer is ***NO***! We can choose combinations of irrational numbers so that we will enter in a cycle and consequently, it's never going to end.
Let us consider $t>1$ and the numbers $M=\left(1,t,t^2,t^3\right)$ for the first line. By performing the operation, we obtain on the second line $\left(t-1,t\left(t-1\right),t^2\left(t-1\right),\left(t^2+t+1\right)\left(t-1\right)\right)$. The equation $t^3=t^2+t+1$ has an irrational solution t_0, as our theoretical survey (cubic equation) says, and, for this value of t_0, it results that the second line is equal with $\left(t_0-1\right)M$. Inductively, the (n+1)th line of the table will be equal with $\left(t_0-1\right)^{n-1}M$,for any natural number *n*.

Line	**Numbers**				
1	**M**	**1**	$\boldsymbol{t_0}$	$\boldsymbol{t_0}^2$	$\boldsymbol{t_0}^3$
2	$\boldsymbol{(t_0-1)}$**M**	$\boldsymbol{t_0-1}$	$\boldsymbol{t_0(t_0-1)}$	$\boldsymbol{t_0^2(t_0-1)}$	$\boldsymbol{(t_0^2+t_0+1)(t_0-1)}$
.	.	.	.	.	.
.	.	.	.	.	.
n+1	$\boldsymbol{(t_0-1)^{n-1}M}$	$\boldsymbol{1(t_0-1)^{n-1}}$	$\boldsymbol{t_0(t_0-1)^{n-1}}$	$\boldsymbol{t_0^2(t_0-1)^{n-1}}$	$\boldsymbol{t_0^3(t_0-1)^{n-1}}$

The result is that we will not get to have a line *n* made up only of zeros.

Theoretical Survey

Cubic Equation

The discriminant of a cubic equation $ax^3+bx^2+cx+d=0$ is given by:
$\Delta_3=b^2c^2-4ac^3-4b^3d-27a^2d^2+18abcd.$
If $\Delta_3<0$, then the equation has one real root and two non-complex conjugate roots.
Regarding our equation, it becomes $t^3-t^2-t-1=0$ with $\Delta=-44<0$, the only real solution is: $t_0\approx 1.8393>1$.

Method of infinite descend

Let *k* be a positive integer. Suppose that whenever *P(m)* holds for some *m>k* then there exists a positive integer *j* such that *m>j>k* and *P(j)* holds. Then *P(n)* is false for all positive integers *n.*
Intuition: If there exists an *n* for which *P(n)* was true, one could construct an infinite sequence $n>n_1>n_2>$... all of which would be greater than *k* but this infinity is impossible.
This technique is known as the *Method of infinite descend.*

Monotony of sequences by induction

Let$a_1=a$, $b_1=b$, $c_1=c$, $d_1=d$ and let $a_n, b_n, c_n, d_n \in$ N, with $a_n=|a_{n-1}-b_{n-1}|$, $b_n=|b_{n-1}-c_{n-1}|$, $c_n=|c_{n-1}-d_{n-1}|$, $d_n=|d_{n-1}-a_{n-1}|$.
Define $x_n=max\{a_n, b_n, c_n, d_n\}$. We apply the induction to prove that x_n is a descending sequence.

Annex 1

```cpp
#include <iostream>
#include <vector>

using namespace std;

vector <int> combinatii_8,copie_init;

int main()
{
    int ok=0;
    int baza2,f,cif,nr_cif,nr_0,primul,ci,nrpasi;
    for (int i=0;i<256;i++)
    {
        baza2=0;
        nr_0=0;
        nr_cif=0;
        f=1;
        ci=i;
        while (ci!=0)
        {
            cif=ci%2;
            baza2=baza2+cif*f;
            f=f*10;
            ci=ci/2;
            nr_cif++;
        }
        for (int j=1;j<=8-nr_cif;j++)
            combinatii_8.push_back(0);

        while (baza2!=0)
        {
            cif=baza2%10;
            combinatii_8.push_back(cif);
            baza2=baza2/10;
        }
        copie_init=combinatii_8;
        do
        {
            primul=combinatii_8[0];
            nr_0=0;
            for (int j=0;j<7;j++)
            {
                if (combinatii_8[j]>combinatii_8[j+1])
                    combinatii_8[j]=combinatii_8[j]-combinatii_8[j+1];
                else
                    combinatii_8[j]=combinatii_8[j+1]-combinatii_8[j];
            }
            if (combinatii_8[7]>primul)
                combinatii_8[7]=combinatii_8[7]-primul;
            else
                combinatii_8[7]=primul-combinatii_8[7];
            for (int j=0;j<8;j++)
                if (combinatii_8[j]==0)
                    nr_0++;
            nrpasi++;

        } while(nr_0<8 && combinatii_8!=copie_init);
        if (nr_0==8)
            ok++;
        else
            i=256;
    }
    if (ok==256)
        cout<<"Conditia este adevarata pentru toate combinatiile."<<endl;
    else
    {
        cout<<"Conditia nu este adevarata pentru toate combinatiile:"<<' ';
        for (int i=0;i<8;i++)
            cout<<combinatii_8[i]<<' ';

    }
    cout<<nrpasi;

    return 0;
}
```

Conclusions:

Our research topic was to prove that we can obtain a null row in these different conditions.
We solved the natural four numbers' case using remainders modulo 2 and we proved that after a certain number of steps, we will reach the zero row: 0, 0, 0, 0.
We also treated the cases of 3, 4, 5, 6, 7, 8, … natural numbers on the first row, obtaining different conclusions, afterwards we made a C++ program.
We extended the case of integers, followed by rational numbers.
Subsequently, we prove significant results for real numbers.
Despite the difficulties we had in solving the problem, it was a challenge for us and an opportunity to improve our knowledge, to discover new fields in mathematics, to make a research work and to write a Mathematical paper, to emphasize one with each other, to develop our teamworking skills, to commit deadlines.
We intend to go further in our research taking into consideration the complex numbers and different applications of the problem in the Cryptography.

References:

”Cubic Equation” - https://www.wolframalpha.com/examples/EquationSolving.html
“Method of infinite descend” - https://proofwiki.org/wiki/Method_of_Infinite_Descent

This article is written by students. It may include omissions and imperfections.

The Lifts

2016- 2017

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Research topic:

This problem is about creating an algorithm that manages 4 lifts in a 10-floor tall building. It takes 3 seconds to get from a floor to another and 15 seconds to stop and take an order.

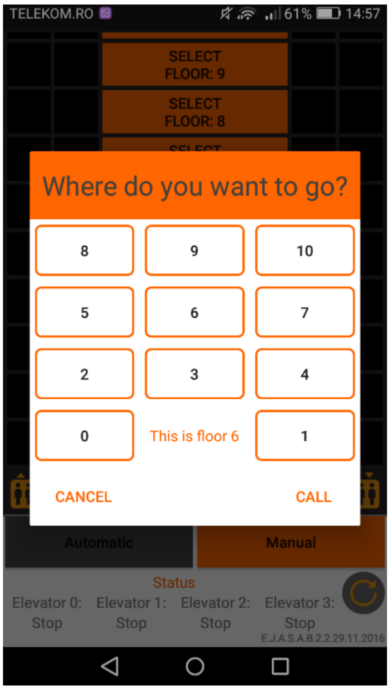
Our work

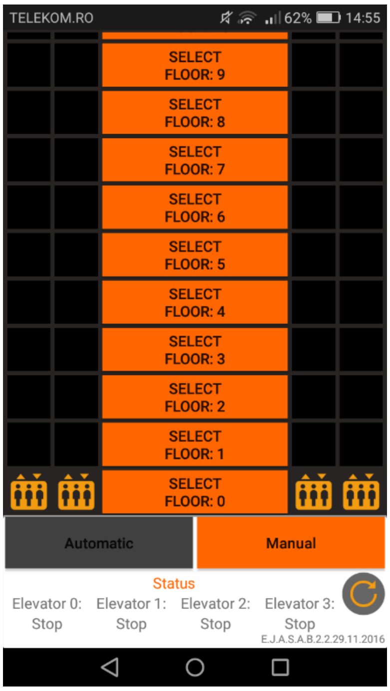
We created an android application that simulates our solution of the problem.

1. Application
2. Solution

1.

From the beginning, we knew that we want to make an Android application and that's how it all started. In the first month, we drew some sketches of the design and of the algorithm that we might use by splitting in 2 teams: one that was thinking the design and one that was searching for the best elevator algorithm possible. We started by programming the menus in the application and after a successful test period we decided that it is time to make the building in which will be the four lifts. We thought of the fastest solution: implementing a custom list of corresponding size in which we will implement the four lifts and one button. After the design was finished and the base functionalities worked we decided to implement the algorithm that we thought of and this was the most challenging part. Every time one elevator moved, we had to refresh the entire list and we had three classes that worked together to get one lift moving. We had to use multiple threads to manage the multiple movements so all was pre-calculated before any visual action and it is checking for the fastest route by three methods and using our formula it gave us the elevator that we can use for the passenger that just called a lift. At every floor, you have a panel that lets you pick your designated level so we know the entire route of a lift before it reaches its destination.





If someone wants to look into our code you can check it out here: https://github.com/Akitektuo/EJAS

2.
One twist we made in order to optimize our times and to come up with a more complex solution was to give more information than the direction where an order wants to go. So, an order will transmit the final destination besides its location.

The first thing we looked into was the condition we have to set in order to avoid possible infinite loops. The measure we took was to let a lift take new orders while delivering a command only if the order goes the same way as the command. For example, if a lift is going down, but on its way down it gets an order that goes up, the lift will ignore the call and keep going, but if the order wants to go down, the lift will stop to take the command, command that we called compatible.

The next thing we done was thinking of how a lift can react when it gets an order. Naturally we are not talking about the first order where the lift will just simply go and complete it. So, after the lift took an order, if it encounters a compatible order on its way to complete the command, it can stop to take the order, or keep going while ignoring the compatible command that it encountered.

The next thing we have to look into is when a lift should ignore or take any new commands. In order to get an answer to this question we created two simple algorithms so we can compare them in different situations. The Direct Algorithm (D.A.) is the algorithm were the lift will not take any new orders encounter on its way and the Basic Algorithm (B.A.) is the algorithm were the lift will stop and take any the order encounter on its way.

The first scenario was a simple one, a call from floor 10 and from floor 9 in the same time to the ground floor. Of course, the D.A. was faster because it does not have to stop to take another order by simply sending two elevators. So, we assumed that the D.A. is faster. The next thing to do was to find when the D.A. is not the fastest, in other words, when the lift should stop and take new orders on its way.

The worst-case scenario was an 8-floor head start (the D.A. has to cover 9 floors while the B.A. has to cover only 1) by asking for an order from floor 10 to the ground floor followed by an order from floor 9 to the ground floor, while the second order is given right after the first order is taken. What is going to happen is that the D.A. will send another lift to take the command from floor 9, while the B.A. will simply take the order and continue its journey. In this case the B.A. is faster than the D.A. by 4.5 seconds – we subtracted the time it took for the B.A. to complete the orders -90 seconds- from the time it took for the D.A. -94.5 seconds- to complete the orders (represented by the average of the 2 lifts, because we care about how much time we have to wait for an order to be completed, so while it’s true that one order took a lot of time to be completed -114 seconds- the other one was completed very fast -75 seconds-).

The next scenario was a 7-floor head start (the D.A has to cover 8 floors while the B.A has to cover 1 floor) by asking for an order from floor 9 to the ground floor followed by an order from floor 8 to the ground floor. The B.A. take 84 seconds to complete the orders while the D.A. takes (69 + 105) / 2 = 87 seconds to complete the orders, which results in a 3 seconds deficit for the D.A. and it also means that the deficit is going down by 1.5 sec by every floor we take from the head start, so if we have a 5-floor head start the times will be equal.

Our solution: A lift will take any new compatible orders on its way unless there is another free lift in a 5-floor range of the floor the order has been made from.

Observation: If there are two elevators which are equally distant from a new call, the one that stopped less time to take orders will go for the call.

This article is written by students. It may include omissions and imperfections.

Winning Bets

2016-2017

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The research topic

At a casino, people are betting on the results of 7 matches. Each match can have 2 outcomes (0-the guest team wins, 1-the away team wins). The first prize is won for guessing all results, the second one for guessing all except one. A playslip consists of the choices of 7 results.

- A: Which is the probability of winning the first prize (respectively the second one) for a randomly completed play-slip?
- B: What is the minimum number of plays-lips which have to be completed in order to surely win the first prize? What is the minimum number of play-slips which have to be completed in order to win at least a first prize or a second prize? Give such an example of completing the play-slips.
- C: Same questions as above but in the hypothesis the matches can have 3 results (0, x,1) and that there are 13 matches (Sweepstakes game).
- D: Generalization.

The conjectures and results obtained

A:

The numbers from the plays-lips have the following form: 1111111, 1101101; 7-digit numbers, the digits can be either 0 or 1.

Therefore, we have in total 2^7 possible plays-lips. So, the probability of winning the first prize with a randomly completed play-slip is 1/128(0.78105%) (favourable cases/total number of cases).

The probability of winning the second prize is 7/128(5.46875%).

We have 7 possibilities of winning the second prize, guessing all the results but one. Being 7 matches there are 7 cases of mistaking the result of one match. For example, if the number 1111111 would be the winning one, we can win the second prize with the following 7 play-slips:

- 0111111
- 1011111
- 1101111
- 1110111
- 1111011
- 1111101
- 1111110

B:

For being sure that we will win the first prize the minimum number of play-slips which we must complete is exactly the number of possible combinations: 128.

Instead, in order to determine the minimum number of play-slips which we have to complete for winning at least a prize (first or second) we have to analyse more carefully the problem.

On the same example: 1111111 we can easily observe that it gives us a possibility of winning the first prize (1111111), and 7 possibilities of winning the second one, the list of the needed numbers was presented in the previous stage of this problem.

So, we deduced the fact that if we complete a play-slip it will take the place, in reality, of 8 completed play-slips, offering a chance of winning the first prize and 7 of winning the second prize.

Therefore, for winning at least a prize we need 128/8=16 completed play-slips. Their filling is, although, difficult, as we have to take into account the following aspect: The chosen numbers can't cover the same number (option) more than once.

For example, if we write the numbers 1111110 and 0111111, we will cover the probability of winning the second prize (if the winning number is 1111111) twice, a fact which has to be avoided. Therefore, we will get to the following conclusion: the chosen numbers has to differ one from the other in at least 3 places. Numbers of the type 111abcd, 000efgh will surely not cover the same second prize option.

The selection of the numbers:

Another part of the problem is and finding the play-slips that will bring us the win. Thus, when we take our example, we have 2 alternative solutions and seven play-slips. So, 128 tickets in total, of which, as already mentioned we choose 16 in order to obtain at least the 2nd prize.

The first step is choosing one of the numbers from the play-slip. Let's pick, for example,0. We divide, then, the numbers in 8 groups, each containing all the numbers that figure 0 as many times as is the number of the group (e.g. Group 3 will contain all the play-slips with 3 variants 0, Group 4 will contain all the play-slips with 4 variants 0, etc).

The next step is to see how many and which positions a play-slip from every group occupies. Let's choose for example the number from group number 3: 0110011. We see this play-slip occupies 3 other play-slips from group number 0 and 4 other play-slips from the group 4. Applying this reasoning for each of the 8 groups we get:

The group number	0	I	II	III	IV	V	VI	VII
0	1	7	0	0	0	0	0	0
I	1	1	6	0	0	0	0	0
II	0	0	1	5	0	0	0	0
III	0	0	3	1	4	0	0	0
IV	0	0	0	4	1	3	0	0
V	0	0	0	0	5	1	0	0
VI	0	0	0	0	0	6	1	1
VII	0	0	0	0	0	0	7	1

In each group, we have the following number of play-slips:

0	1 (C_7^0)
I	7 (C_7^1)
II	21 (C_7^2)
III	35 (C_7^3)
IV	35(C_7^4)
V	21(C_7^5)
VI	7 (C_7^6)
VII	1 (C_7^7)

In order for every play-slip to be covered by the other chosen play-slip, we have to be careful at the distribution of the 16 play-slips in the 8 groups. Because group number 7 can be covered by play-slips from groups 6 and 7, the first 2 play-slips chosen will be from the groups 7 and 0 (because of the symmetry): 0000000, 111111. Choosing these ones will have covered all the play-slips from the 1 and 6 groups.

We observe that the groups 4 and 5 have 35 and 21 play-slips, which both are multiples of 7. Thus, in order to make sure that all the play-slips from groups 5 and 2 will be covered will choose 7 from the fourth group and 7 from the third one. In total, we will have 16 play-slips from all the groups:

Group number	0	I	II	III	IV	V	VI	VII
0	1	7	0	0	0	0	0	0
III	0	0	21	7	28	0	0	0
IV	0	0	0	28	7	21	0	0
VII	0	0	0	0	0	0	7	1
Total	1	7	21	35	35	21	7	1

We saw from which groups we have to choose the numbers. Now we have to see which are the actual numbers. Let's analyse the third group. Each play-slip covers four other play-slips from the fourth group. The condition is that the chosen play-slips can't cover the same thing. So, they have to be different in 4 positions. Applying this condition, we have the numbers from the third group:

- 0011101; 0101110;
- 0110011; 1001011;
- 1010110; 1100101;
- 1111000.

We will choose the numbers from the fourth group by symmetry.

In conclusion, the numbers chosen are:

- 0011101; 0101110;
- 0110011; 1001011;
- 1010110; 1100101;
- 1111000; 1100010;
- 1010001; 1001100;
- 0110100; 0101001;
- 0011010; 0000111;
- 1111111; 0000000.

C:

The probability of winning the first prize it's the number of favourable cases divided by the number of all cases. So, it's $1/3^{13}$ (0.0000607%). For the second prize, the chance is 26 (number of possible cases)/3^{13} (0.0016307%). Now, for completing one play-slip randomly we have one possibility of winning the first prize and 26 possibilities of winning the second prize. For example, if the winning number is 1111111111111 we can win the first prize by completing this number and second prize by completing any of the following numbers:

- 0111111111111 x111111111111
- 1011111111111 1x11111111111
- 1101111111111 11x1111111111
- 1110111111111 111x111111111
- 1111011111111 1111x11111111
- 1111101111111 11111x1111111
- 1111110111111 111111x111111
- 1111111011111 1111111x11111
- 1111111101111 11111111x1111
- 1111111110111 111111111x111

- 1111111111011 1111111111x11
- 1111111111101 11111111111x1
- 1111111111110 111111111111x

Thus, the minimum number for winning surely one prize is the number of all possible cases/the number of cases in which you win the first prize(1) + the number of cases which you win the second prize(26). So, it's $3^{13}/27=3^{10}$.

D:

We will define the number of the matches in a play-slip with "n" and with x the number of possible outcomes of a match.

Therefore, the total number of possible play-slips will be x^n. The chance of winning the first prize is $1/x^n$.(and for winning surely the first prize we have to complete obviously x^n.

Now, a completed play-slip offers us a chance of winning the first prize, and (x-1)*n possibilities of winning the second prize. So, for a chance of winning the first or the second prize we have to complete:

$$\frac{x^n}{(x-1)*n+1}$$

Conclusions:

- Each play-slip can cover other play-slips that can bring lower prizes;
- If we would have "x" outcomes for a match and "n" matches we would need $\frac{x^n}{(x-1)*n+1}$ completed playslips to be sure we would win any of the first two prizes;
- If we want to find the lucky play-slips we need to group all the possible ones and choose the calculated number of play-slips so that they will not cover the same play-slip.

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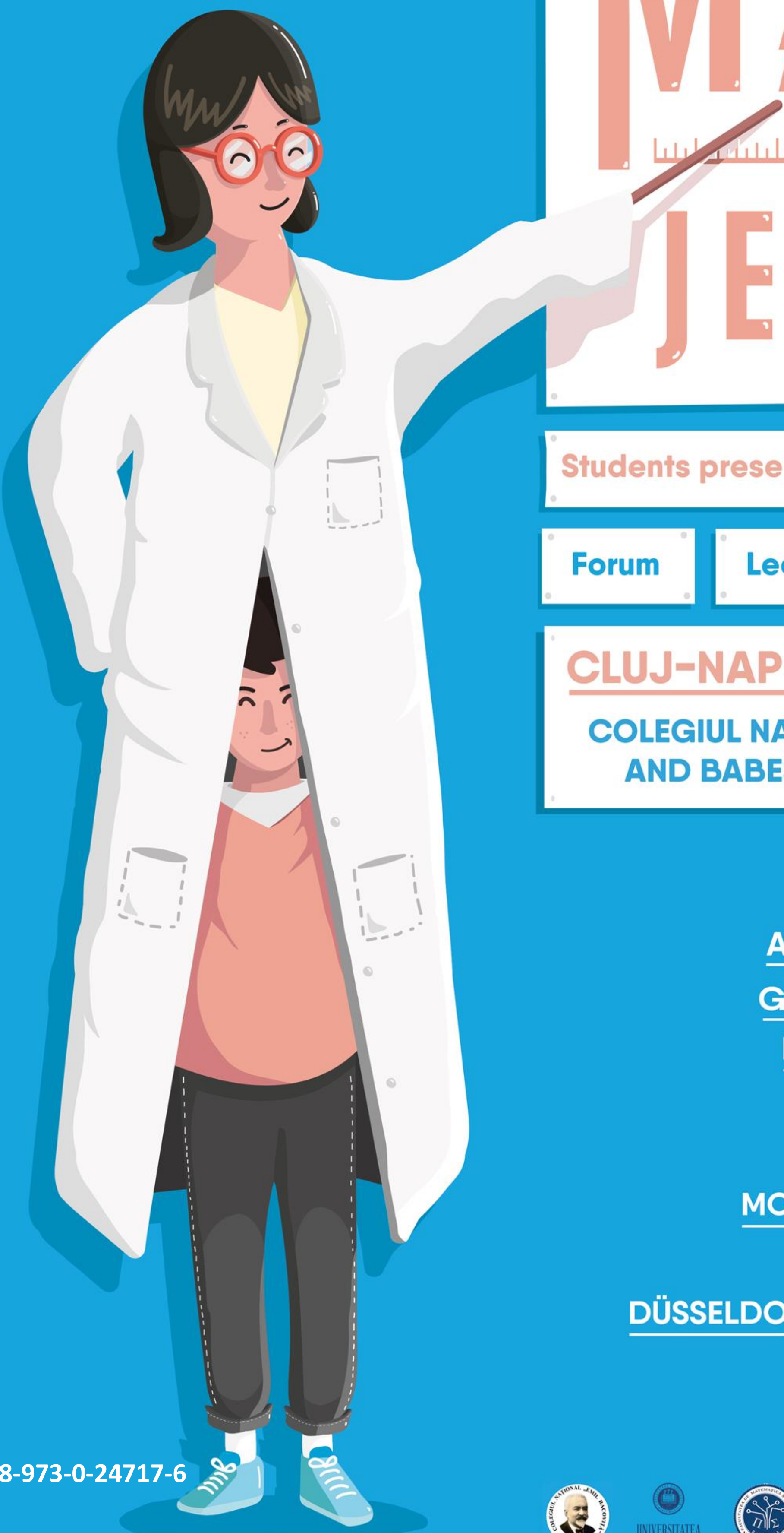

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